Hyperbolic Functions

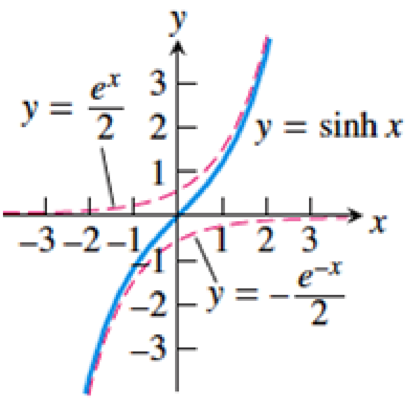
The last set of functions that we're going to be looking are the hyperbolic functions. In many physical situations combinations of e^x and e^{-x} arise fairly often. Because of this these combinations are given names. There are six hyperbolic functions and they are defined as follows

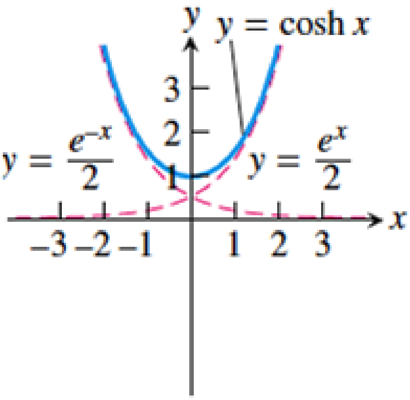
$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

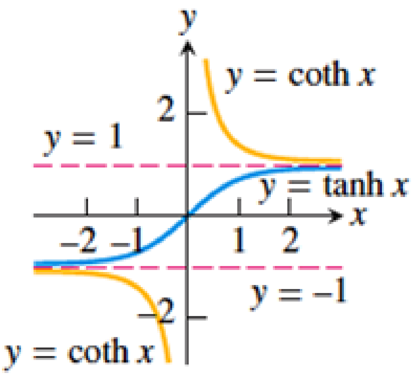
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

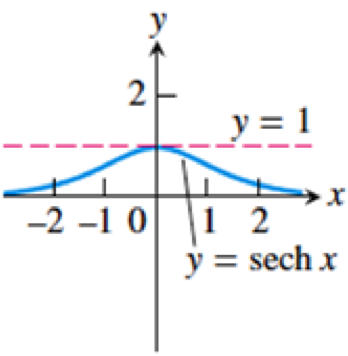
$$\coth x = \frac{1}{\tanh x}, \quad \operatorname{sech} x = \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x}$$

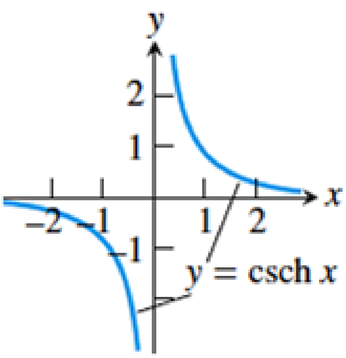
Graphs of Hyperbolic Functions











Some Identities for Hyperbolic Functions

We have the following facts about the hyperbolic functions

1. $\cosh x \geq 1, \quad 0 < \operatorname{sech} x \leq 1$ and $-1 < \tanh x < 1$ for all x
2. $\sinh(-x) = -\sinh(x)$ and $\cosh(-x) = \cosh(x)$
3. $\sinh(0) = 0$ and $\cosh(0) = 1$
4. $\cosh^2 x - \sinh^2 x = 1$
5. $\sinh 2x = 2 \sinh x \cosh x$
6. $\cosh 2x = \cosh^2 x + \sinh^2 x$
7. $\sinh x + \cosh x = e^x$
8. $\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$
9. $\cosh(x+y) = \cosh x \cosh y + \sinh y \sinh x$

Proof

4. $$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$
$$= \frac{e^{2x} + 2e^{x-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^{x-x} + e^{-2x}}{4}$$
$$= \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4}$$
$$= \frac{4}{4} = 1$$

8. $$\sinh x \cosh y + \sinh y \cosh x = \frac{e^x - e^{-x}}{2} \times \frac{e^y + e^{-y}}{2} + \frac{e^y - e^{-y}}{2} \times \frac{e^x + e^{-x}}{2}$$
$$= \frac{e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y} + e^{x+y} + e^{y-x} - e^{-y+x} - e^{-x-y}}{4}$$
$$= \frac{2e^{x+y} - 2e^{-x-y}}{4} = \frac{2\left(e^{x+y} - e^{-(x+y)}\right)}{4}$$
$$= \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

Solving Equations

Example1: Find the value of x for which $\sinh x = 0.75$

$$\frac{e^x - e^{-x}}{2} = \frac{3}{4} \quad \Rightarrow \quad \{ \quad 2e^x - 2e^{-x} = 3 \quad \} \times e^x$$

$$2e^{2x} - 3e^x - 2 = 0 \quad \Rightarrow \quad (2e^x + 1)(e^x - 2) = 0$$

$$e^x = 2 > 0 \quad \text{or} \quad e^x = -\frac{1}{2} < 0$$

But $e^x > 0 \ \forall x \in R$, so $e^x = 2 \ \Rightarrow \ x = \ln 2 = 0.693$

We can solve this example by another way:

$$\sinh x = \frac{3}{4}$$

But, $\cosh^2 x = 1 + \sinh^2 x$

$$\cosh^2 x = 1 + \frac{9}{16} = \frac{25}{16} \quad \Rightarrow \quad \cosh x = \frac{5}{4} \quad (\cosh x \geq 1 \ \ \forall x \in R)$$

$$\sinh x + \cosh x = e^x \quad \Rightarrow \quad e^x = \frac{3}{4} + \frac{5}{4} = 2$$

$$x = \ln 2 = 0.693$$

Example 2: Solve for x: $2\sinh x - 3\cosh x = -3$

$$2\left(\frac{e^x - e^{-x}}{2}\right) - 3\left(\frac{e^x + e^{-x}}{2}\right) = -3$$

$$\frac{2e^x - 2e^{-x} - 3e^x - 3e^{-x}}{2} = -3$$

$$-e^x - 5e^{-x} = -6$$

$$\{e^x + 5e^{-x} = 6\} \times e^x$$

$$e^{2x} - 6e^x + 5 = 0$$

$$(e^x - 1)(e^x - 5) = 0$$

$$e^x = 1 \quad \text{or} \quad e^x = 5$$

$$x = \ln 1 = 0 \quad \text{or} \quad x = \ln 5 = 1.609$$

Example 3: Solve for x: $5\cosh x + 3\sinh x = 4$

$$5\left(\frac{e^{x}+e^{-x}}{2}\right)+3\left(\frac{e^{x}-e^{-x}}{2}\right)=4$$

$$\frac{5e^{x}+5e^{-x}+3e^{x}-3e^{-x}}{2}=4$$

$$\frac{8e^{x}+2e^{-x}}{2}=4$$

$$4e^{x}+e^{-x}=4$$

$$4e^{x}+\frac{1}{e^{x}}=4$$

$$\frac{4e^{2x}+1}{e^{x}}=4$$

$$4e^{2x}+1=4e^{x}$$

$$4(e^{x})^{2}-4e^{x}+1=0$$

$$(2e^{x}-1)^{2}=0$$

$$e^{x}=1/2$$

$$x=\ln(1/2)=-\ln 2=-0.693$$

Derivatives of Hyperbolic Functions

If $u = u(x)$, then:

1. $\dfrac{d}{dx}(\sinh u) = \cosh u \dfrac{du}{dx}$

2. $\dfrac{d}{dx}(\cosh u) = \sinh u \dfrac{du}{dx}$

3. $\dfrac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \dfrac{du}{dx}$

4. $\dfrac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \dfrac{du}{dx}$

5. $\dfrac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \dfrac{du}{dx}$

6. $\dfrac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \dfrac{du}{dx}$

Example 4: Find derivatives of the functions

1. $y = \cosh(x^2 - 3)$ ⇨ $\dfrac{dy}{dx} = 2x \sinh(x^2 - 3)$
2. $y = \sinh\sqrt{x^2+1}$ ⇨ $\dfrac{dy}{dx} = \cosh\sqrt{x^2+1} \times \dfrac{2x}{2\sqrt{x^2+1}} = \dfrac{x\cosh\sqrt{x^2+1}}{\sqrt{x^2+1}}$
3. $y = \tanh\sqrt{x}$ ⇨ $\dfrac{dy}{dx} = \operatorname{sech}^2\sqrt{x} \times \dfrac{1}{2\sqrt{x}} = \dfrac{\operatorname{sech}^2\sqrt{x}}{2\sqrt{x}}$
4. $y = e^{2x}\cosh 3x$ ⇨ $\dfrac{dy}{dx} = 3e^{2x}\sinh 3x + 2e^{2x}\cosh 3x$

Exercises

1. Solve for $x \in R$

$a.$ $4\cosh x + \sinh x = 4$ $b.$ $3\sinh x - \cosh x = 1$

$c.$ $\operatorname{sech} 2x = 0.25$ $d.$ $10\sinh 2x + 2\cosh 2x = 5$

2. Find derivatives of the functions

$a.$ $y = \cosh 2x - \sinh 3x$ $b.$ $y = e^{-2x} \sinh 2x$

$c.$ $y = 3x \operatorname{sech} 2x$ $d.$ $y = \sqrt{x}\tanh\sqrt{x}$