

Multiple Integrals

The multiple integral is a generalization of the definite integral to functions of more than one real variable. Integrals of a function of two variables are called double integrals, and integrals of a function of three variables are called triple integrals.

Double Integrals

The expression $\int_c^d \int_a^b f(x, y) dx dy$ is called double integral and indicates that:

1. $f(x, y)$ is first integrated with respect to x (regarding y as being constant) between the limits $x = a$ and $x = b$.
2. the result is then integrated with respect to y between the limits $y = c$ and $y = d$.

Example 1: Evaluate $\int_1^2 \int_2^4 (x + 2y) dx dy$

$$\begin{aligned} \int_1^2 \int_2^4 (x + 2y) dx dy &= \int_1^2 \left[\frac{x^2}{2} + 2yx \right]_2^4 dy = \int_1^2 (8 + 8y - 2 - 4y) dy \\ &= \int_1^2 (4y + 6) dy = \left[2y^2 + 6y \right]_1^2 = 8 + 12 - 2 - 6 = 12 \end{aligned}$$

Example 2: Evaluate $\int_0^{\pi} \int_0^{\sin x} y dy dx$

$$\begin{aligned} \int_0^{\pi} \int_0^{\sin x} y dy dx &= \int_0^{\pi} \left[\frac{y^2}{2} \right]_0^{\sin x} dx = \int_0^{\pi} \frac{\sin^2 x}{2} dx = \int_0^{\pi} \frac{1 - \cos 2x}{4} dx \\ &= \left[\frac{1}{4}x - \frac{1}{8}\sin 2x \right]_0^{\pi} = \frac{\pi}{4} \end{aligned}$$

Example 3: Evaluate $\int_1^2 \int_y^{y^2} dx dy$

$$\int_1^2 \int_y^{y^2} dx dy = \int_1^2 x \Big|_y^{y^2} dy = \int_1^2 (y^2 - y) dy = \frac{y^3}{3} - \frac{y^2}{2} \Big|_1^2 = \frac{5}{6}$$

Example 4: Evaluate $\int_0^1 \int_0^x \sqrt{1-x^2} dy dx$

$$\begin{aligned} \int_0^1 \int_0^x \sqrt{1-x^2} dy dx &= \int_0^1 y \Big|_0^x \sqrt{1-x^2} dx = \int_0^1 x \sqrt{1-x^2} dx = \frac{-1}{2} \int_0^1 -2x(1-x^2)^{1/2} dx \\ &= -\frac{1}{2} (1-x^2)^{3/2} \times \frac{2}{3} \Big|_0^1 = -\frac{1}{3} (0-1) = \frac{1}{3} \end{aligned}$$

Triple Integrals

It will come as no surprise that we can also do triple integrals

Example 5: Evaluate $\int_0^2 \int_{-1}^1 \int_0^1 (2x - y + z) dx dy dz$

$$\begin{aligned} \int_0^2 \int_{-1}^1 \int_0^1 (2x - y + z) dx dy dz &= \int_0^2 \int_{-1}^1 (x^2 - yx + zx) \Big|_0^1 dy dz = \int_0^2 \int_{-1}^1 (1 - y + z) dy dz \\ &= \int_0^2 \left(y - \frac{y^2}{2} + zy \right) \Big|_{-1}^1 dz = \int_0^2 \left(1 - \frac{1}{2} + z - \left(-1 - \frac{1}{2} - z \right) \right) dz \\ &= \int_0^2 (2 + 2z) dz = 2z + z^2 \Big|_0^2 = 8 \end{aligned}$$

Example 6: Evaluate $\int_0^\pi \int_0^\pi \int_0^3 x^2 \sin \theta dx d\theta d\phi$

$$\int_0^\pi \int_0^\pi \int_0^3 x^2 \sin \theta dx d\theta d\phi = \int_0^\pi \int_0^\pi \left(\frac{x^3}{3} \Big|_0^3 \right) \sin \theta d\theta d\phi = \int_0^\pi \int_0^\pi 9 \sin \theta d\theta d\phi$$

$$= \int_0^{\pi} -9 \cos \theta \left| d\theta \right| = \int_0^{\pi} -9(-1 - 1) d\theta = \int_0^{\pi} 18 d\theta = 18 \theta \Big|_0^{\pi} = 18\pi$$

Exercises

Evaluate each of the following integrals

1. $\int_0^2 \int_1^{e^x} dy dx$

2. $\int_0^1 \int_{\sqrt{y}}^1 dx dy$

3. $\int_{-2}^1 \int_{x^2+4x}^{3x+2} dy dx$

4. $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy dx dy$

5. $\int_0^1 \int_{x^2}^1 (x + y) dy dx$

6. $\int_0^1 \int_0^{y^2} \sqrt{y^3 + 3} dx dy$

7. $\int_0^1 \int_0^2 x \sqrt{4 - x^2} dx dy$

8. $\int_0^2 \int_1^3 \int_1^2 xy^2 z dx dy dz$

9. $\int_0^2 \int_0^1 \int_0^1 (x^2 + y^2 + z^2) dx dy dz$

10. $\int_0^2 \int_{3x/2}^3 \int_0^{5x/2} dz dy dx$