CHAPTER FIVE: DIGITAL FILTER DESIGN

1. Structures for IIR Systems

1.1 Direct Form I

The input $x(n)$ and output $y(n)$ of a causal (Infinite Impulse Response) IIR filter with a rational system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{1.1}$$

is described by the linear constant coefficient difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \tag{1.2}$$

or,

$$y(n) = \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \tag{1.3}$$

The block diagram of Fig.(1.1) is an explicit pictorial representation of Eq.(1.3). More precisely, it represents the pair of difference equations

$$v(n) = \sum_{k=0}^{M} b_k x(n-k) \tag{1.4a}$$

$$y(n) = v(n) - \sum_{k=1}^{N} a_k y(n-k) \tag{1.4b}$$

From Eqn.(1.1), Fig.(1.1) can be viewed as an implementation of $H(z)$ through the decomposition

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right)\left(\sum_{k=0}^{M} b_k z^{-k}\right) \tag{1.5}$$

or, equivalently, through the pair of equations

$$V(z) = H_1(z)X(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) X(z) \tag{1.6a}$$

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}\right) V(z) \tag{1.6b}$$

Figure (1.1) can be viewed as a cascade of two systems, the first representing the computation of $v(n)$ from $x(n)$ and the second representing the computation of $y(n)$ from $v(n)$.

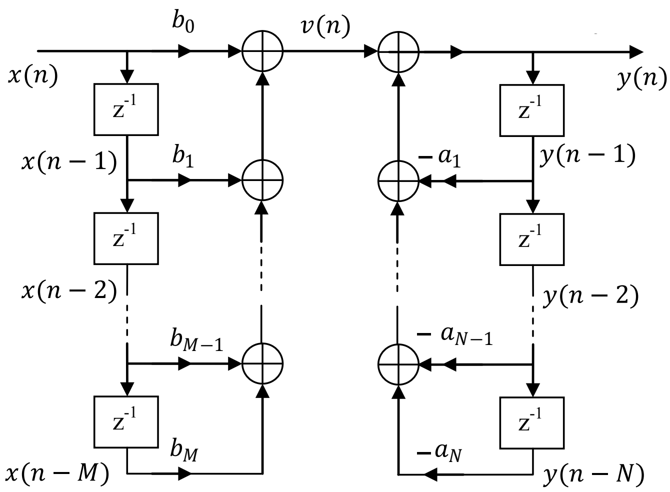


Fig.(1.1)

1.2 Direct Form II

Since each of the two systems, $H_1(z)$ and $H_2(z)$ is a linear time-invariant system, the order in which the two systems are cascaded can be reversed, as shown in Fig.(1.2), without affecting the overall system function. For convenience, we have assumed that $M = N$.

$$H(z) = H_1(z)H_2(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right)\left(\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}\right) \tag{1.7}$$

or, equivalently, through the pair of equations

$$W(z) = H_2(z)X(z) = \left(\frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}\right)X(z) \tag{1.8a}$$

$$Y(z) = H_1(z)W(z) = \left(\sum_{k=0}^{M} b_k z^{-k}\right) W(z) \tag{1.8b}$$

In the time domain, Fig.(1.2) and, equivalently, Eqn.(1.8) can be represented by the pair of difference equations

$$w(n) = x(n) - \sum_{k=1}^{N} a_k w(n-k) \tag{1.9a}$$

$$y(n) = \sum_{k=0}^{M} b_k w(n-k) \tag{1.9b}$$

The systems in Fig.(1.1) and (1.2) each have a total of $(N + M)$ delay elements. However, the block diagram of Fig.(1.2) can be redrawn by noting that exactly the same signal, $w(n)$, is stored in the two chains of delay elements in the figure. Consequently, the two can be collapsed into one chain, as indicated in Fig.(1.3).

The total number of delay elements in Fig.(1.3) is less than in either Fig.(1.1) or Fig.(1.2). Specifically, the minimum number of delays required is, in general, $max(N, M)$.

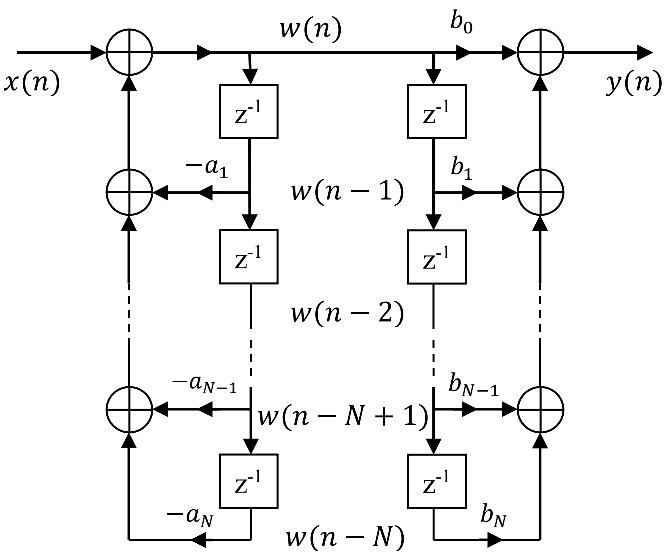


Fig.(1.2)

Example 1.1: Draw the block diagram and the signal flow graph using direct form I and II realization of the discrete-time system represented by the transfer function

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

Solution: Comparing this system function with Eqn.(1.1), we find $b_0 = 1$, $b_1 = 2$, $a_1 = -1.5$, and $a_2 = 0.9$. Figure (1.4a) and (1.4b) depict a pictorial diagram for the direct form II and I, respectively. Figure (1.4a) and (1.4b) can be rewritten applying the signal flow graph, as shown in Fig.(1.5a) and (1.5b)

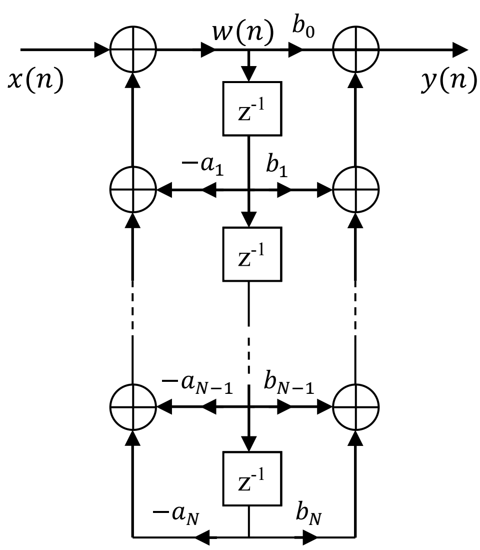
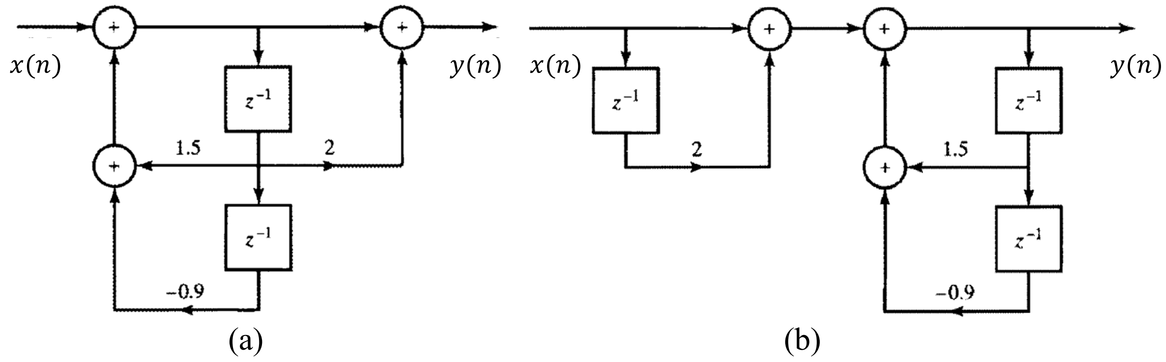
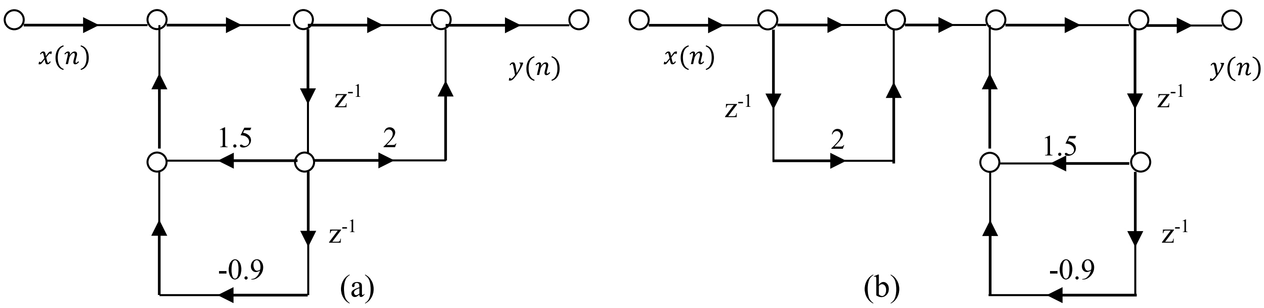


Fig.(1.3)



Fig(1.4)



Fig(1.5)

H.W: Draw the block diagram and the signal flow graph using direct form I and II realization of the discrete-time system represented by the transfer function

$$H(z) = \frac{8z^3 - 4z^2 + 11z - 2}{\left(z - \frac{1}{4}\right)\left(z^2 - z + \frac{1}{2}\right)}$$

1.3 Cascade Structure

The cascade structure is derived by factoring the numerator and denominator polynomials of $H(z)$:

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = A \prod_{k=1}^{\max\{N,M\}} \frac{1 - \beta_k z^{-1}}{1 - \alpha_k z^{-1}} \tag{1.10}$$

This factorization corresponds to a *cascade* of first-order filters, each having one pole and one zero. In general the coefficients α_k and β_k will be complex. However, if $h(n)$ is real, the roots of $H(z)$ will occur in complex conjugate pairs, and these complex conjugate factors may be combined to form second-order factors with *real* coefficients:

$$H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

There is considerable flexibility in how a system may be implemented in cascade form. For example, there are different *pairings* of the poles and zeros and different ways in which the sections may be *ordered.* For example the system

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

has a direct form I and direct form II structures shown in Fig.(1.6)

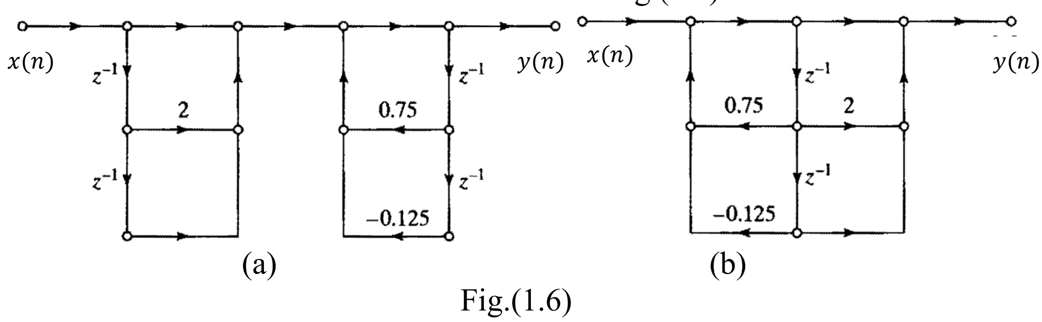


Fig.(1.6)

Alternatively, to illustrate the cascade structure, we can use first-order systems by expressing $H(z)$ as a product of first-order factors, as in

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

Since all of the poles and zeros are real, a cascade structure with first-order sections has real coefficients. If the poles and/or zeros were complex, only a second-order section would have real coefficients. Fig.(1.7) shows two equivalent cascade structures.

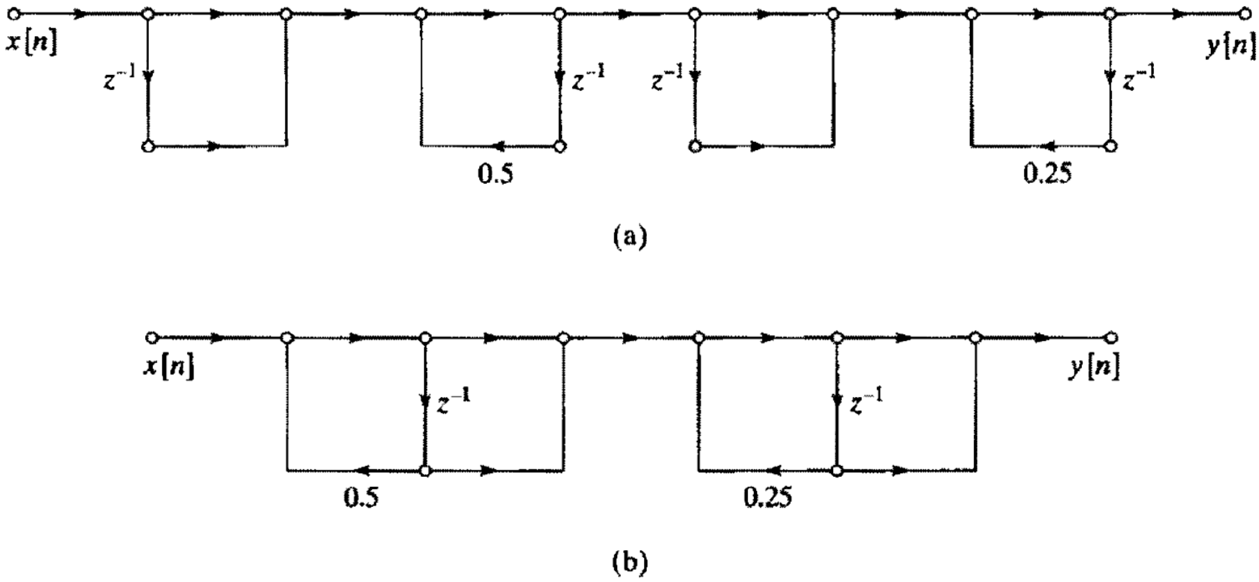


Fig.(1.7): (a) Direct form I subsections.(b) Direct form II subsections.

1.4 Parallel Structure

An alternative to factoring $H(z)$ is to expand the system function using a partial fraction expansion. For example, with

$$H(z) = A\frac{\prod_{k=1}^{M}(1 - \beta_k z^{-1})}{\prod_{k=1}^{N}(1 - \alpha_k z^{-1})}$$

If $N > M$ and $\alpha_i \neq \alpha_k$ (the roots of the denominator polynomial are distinct), $H(z)$ may be expanded as a sum of ***N*** first-order factors as follows:

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - \alpha_k z^{-1}}$$

where the coefficients A_k and α_k are, in general, complex. This expansion corresponds to a sum of *N* first-order system functions and may be realized by connecting these systems in parallel. If $h(n)$ is real, the poles of $H(z)$ will occur in complex conjugate pairs, and these complex roots in the partial fraction expansion may be combined to form second-order systems with real coefficients:

$$H(z) = \sum_{k=1}^{N_s} \frac{\gamma_{0k} + \gamma_{1k} z^{-1}}{1 + \alpha_{1k} z^{-1} + \alpha_{2k} z^{-2}}$$

Shown in Fig.(1.8) is a sixth-order filter implemented as a parallel connection of three second-order direct form II systems. If $N \leq M$, the partial fraction expansion will also contain a term of the form

$$c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$$

which is an FIR filter that is placed in parallel with the other terms in the expansion of $H(z)$.

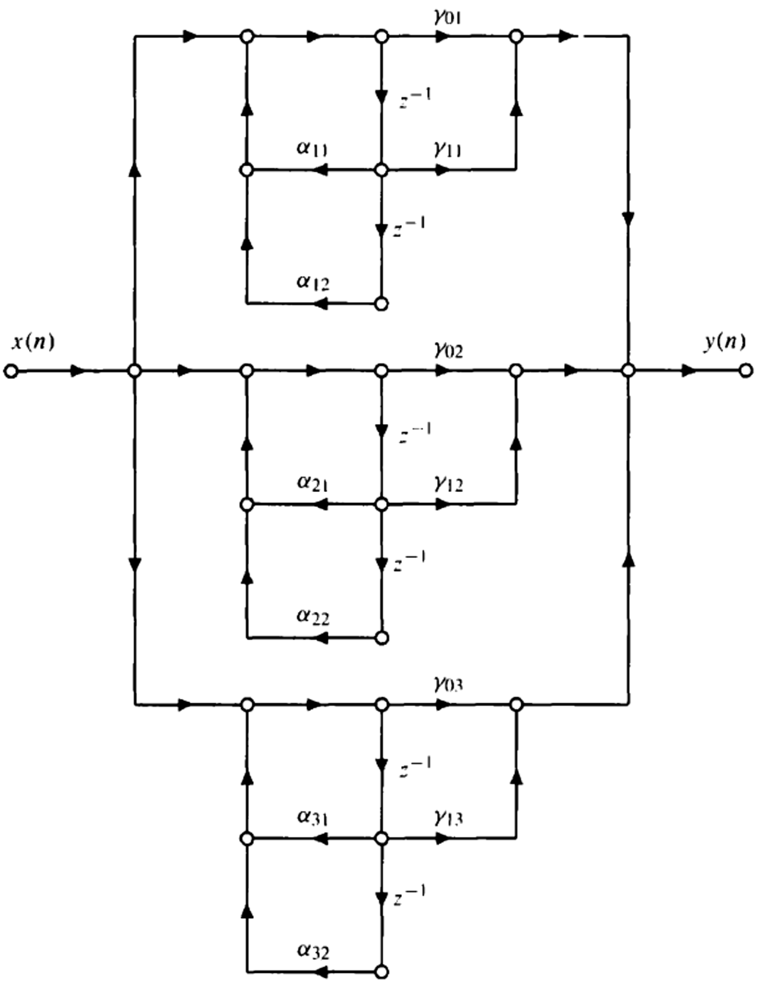


Fig.(1.8)

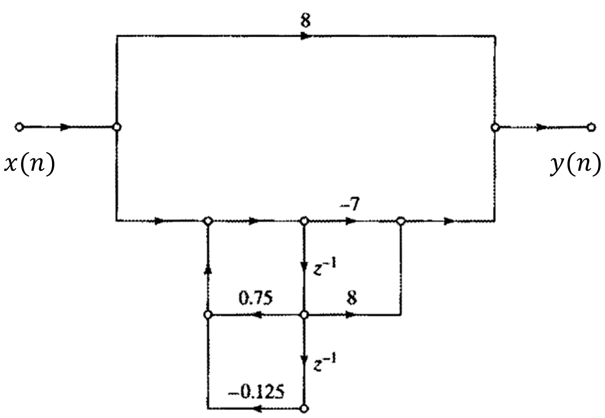
The parallel-form realization for the system with a second-order section is shown in Fig.(1.9a).

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Since all the poles are real, we can obtain an alternative parallel form realization by expanding $H(z)$ as

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}}$$

The resulting parallel form with first-order sections is shown in Fig.(1.9b).



(a)

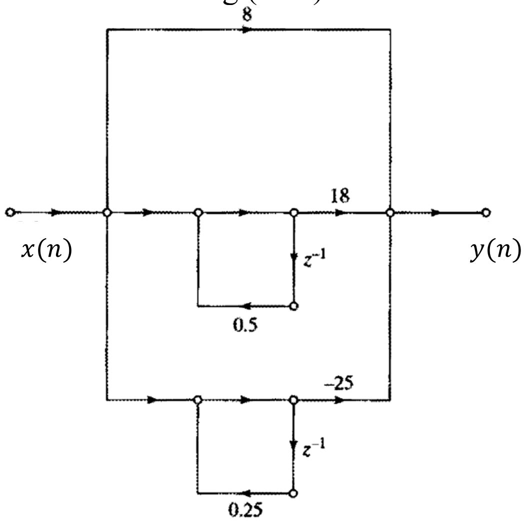


Fig.(1.9) (b)

2. Structures for FIR Systems

A causal FIR filter has a system function that is a polynomial in z^{-1}:

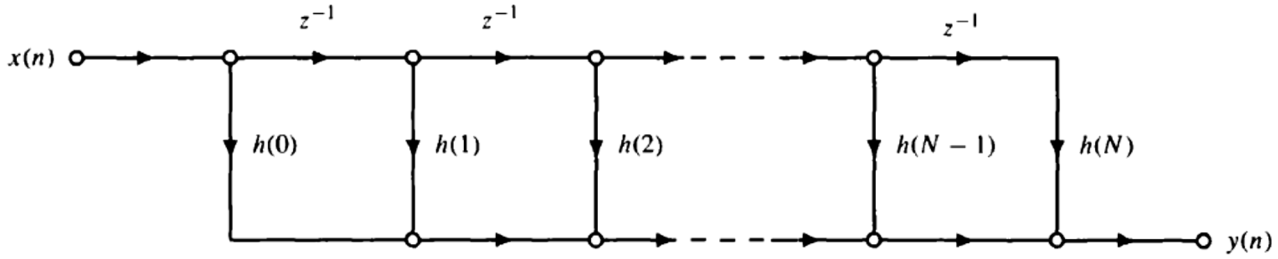
$$H(z) = \sum_{n=0}^{N} h(n)z^{-n}$$

For an input $x(n)$, the output is

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

2.1 Direct Form

The most common way to implement an FIR filter is in direct from using a tapped delay line as shown in the figure below



2.2 Cascade Form

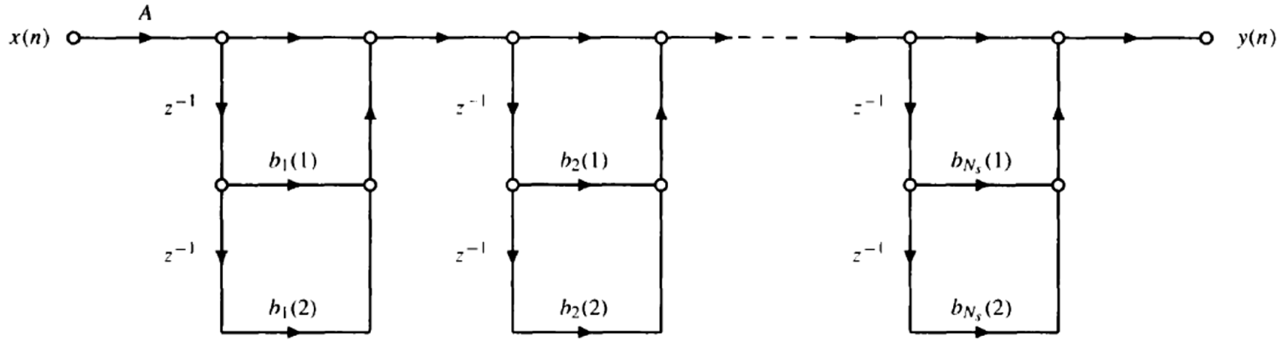
For a causal FIR filter, the system function may be factored into a product of first-order factors,

$$H(z) = \sum_{n=0}^{N} h(n)z^{-n} = A\prod_{k=1}^{N}(1-\alpha_k z^{-1})$$

where α_k for $k = 1,\dots,N$ are the zeros of $H(z)$. If $h(n)$ is real, the complex roots of $H(z)$ occur in complex conjugate pairs, and these conjugate pairs may be combined to form second-order factors with real coefficients,

$$H(z) = A\prod_{k=1}^{N_S}[1 + b_k(1)z^{-1} + b_k(2)z^{-2}]$$

$H(z)$ may be implemented as a cascade of second-order FIR filters as illustrated in Figure below.



3. IIR FILTER DESIGN

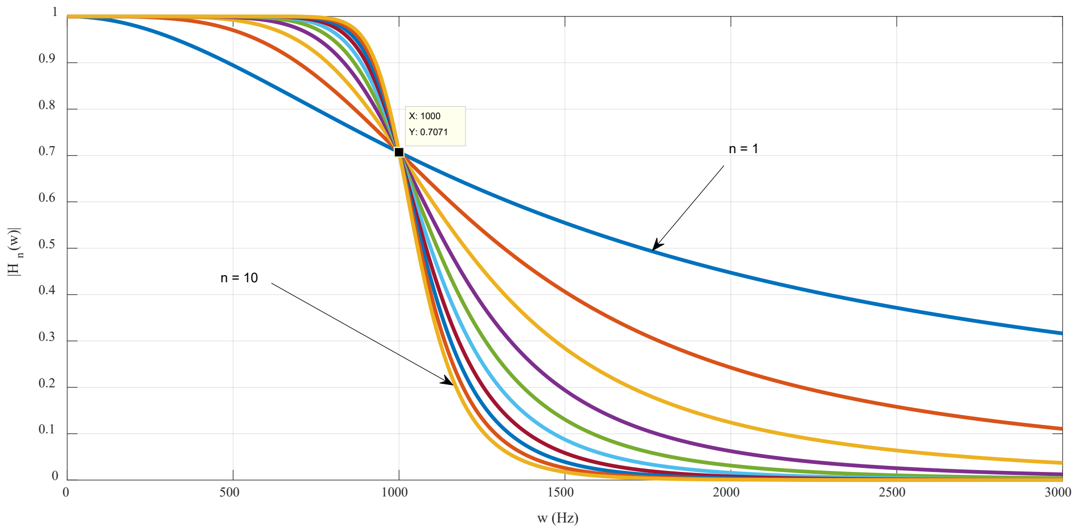
3.1 Butterworth Filters

A unity-gain Butterworth low-pass filter has a transfer function whose magnitude is given by

$$|H_n(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}} \qquad (3.1)$$

Where *n* is an integer that denotes the order of the filter.

1. The cutoff frequency is Ω_c rad/s for all values of *n*.
2. If *n* is large enough, the denominator is always close to unity when $\Omega < \Omega_c$
3. In the expression for $|H_n(\Omega)|$, the exponent of (Ω/Ω_c) is always even.



To derive $H(s)$, let us set $\Omega_c = 1\ rad/s$ (prototype filter), and note that

$$|H_n(j\Omega)|^2 = H_n(j\Omega)H_n(-j\Omega) = \frac{1}{1+\Omega^{2n}}$$

But because $s = j\Omega$, we can write

$$|H_n(s)|^2 = H_n(s)H_n(-s)$$

Thus,

$$|H_n(s)|^2 = \frac{1}{1+(s/j)^{2n}}$$

The procedure for finding $H_n(s)$ for a given value of n is as follows:
1. Find the roots of the polynomial

$$1+(s/j)^{2n} = 0$$

or

$$s^{2n} = -1(j)^{2n} = (-1)^{n+1}$$

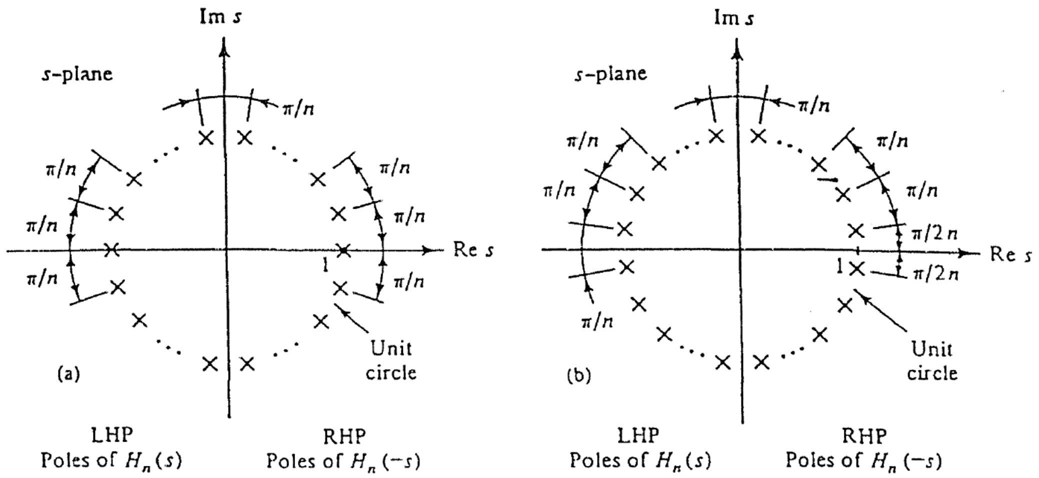
for n odd: $s_k = 1\angle k\pi/n,\qquad k = 0, 1, 2, \ldots, 2n-1$

for n even: $s_k = 1\angle \pi/2n + k\pi/n,\qquad k = 0, 1, 2, \ldots, 2n-1$

2. Assign the left-half plane roots to $H_n(s)$ and the right-half plane roots to $H_n(-s)$.
3. Combine terms in the denominator of $H_n(s)$ to form first- and second-order factors.

$H_n(s)$ can be written in the following form:

$$H_n(s) = \frac{1}{\prod_{\text{LHP poles}}(s-s_k)} = \frac{1}{B_n(s)} \qquad (3.2)$$



For n odd

For n even

Table (3.1): Butterworth Polynomials in Standard and Factored Forms

Standard form

$$B_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$$

a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0	n
							1	1	1
						1	$\sqrt{2}$	1	2
					1	2	2	1	3
				1	2.613	3.414	2.613	1	4
			1	3.236	5.236	5.236	3.236	1	5
		1	3.864	7.464	9.141	7.464	3.864	1	6
	1	4.494	10.103	14.606	14.606	10.103	4.494	1	7
1	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1	8

Factored form

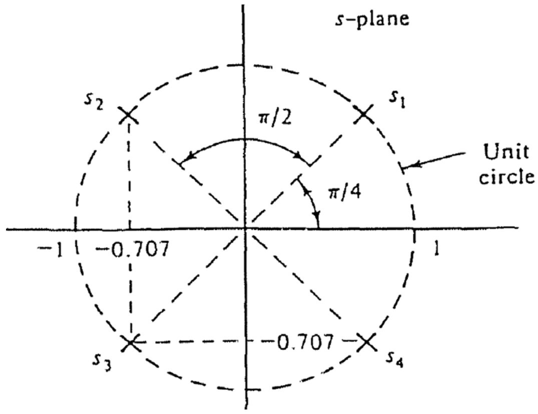
$B_n(s)$	n
$s + 1$	1
$s^2 + \sqrt{2}s + 1$	2
$(s^2 + s + 1)(s + 1)$	3
$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$	4
$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$	5
$(s^2 + 0.5176s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 1.9318s + 1)$	6
$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2456s + 1)(s^2 + 1.8022s + 1)$	7
$(s^2 + 0.3986s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)(s^2 + 1.9622s + 1)$	8

Butterworth filter

$$H_n(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} = \frac{1}{B_n(s)}$$

Example 3.1: Find the transfer function $H_2(s)$ for the normalized Butterworth filter of order 2.
Solution: Since $n = 2$ we have the poles of $H_2(s)H_2(-s)$ given by

$$s_k = 1\angle \pi/4 + k\pi/2, \qquad k = 0, 1, 2, 3$$



Therefore, the four roots are

$s_1 = 1\angle 45^\circ = 1/\sqrt{2} + j/\sqrt{2},$

$s_2 = 1\angle 135^\circ = -1/\sqrt{2} + j/\sqrt{2},$

$s_3 = 1\angle 225^\circ = -1/\sqrt{2} + -j/\sqrt{2},$

$s_4 = 1\angle 315^\circ = 1/\sqrt{2} + -j/\sqrt{2}.$

Using the left-half plane poles we can express the transfer function as follows

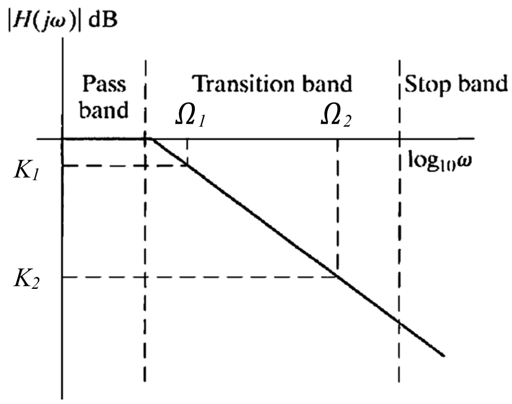
$$H_2(s) = \frac{1}{(s - s_2)(s - s_3)}$$

$$= \frac{1}{[s - (-0.707 - 0.707j)][s - (-0.707 + 0.707j)]}$$

$$= \frac{1}{s^2 + \sqrt{2}s + 1}$$

3.2 The Order of a Butterworth Filter

In the design of a low-pass filter, the filtering specifications are usually given in terms of the abruptness of the transition region, as shown in Figure beside. Once K_1, Ω_1, K_2, and Ω_2 are specified, the order of the Butterworth filter can be determined. For the Butterworth filter,



$$K_1 = 20log_{10}\frac{1}{\sqrt{1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n}}} = -10log_{10}\left(1+\left(\frac{\Omega_1}{\Omega_c}\right)^{2n}\right) \quad (3.3)$$

$$K_2 = 20log_{10}\frac{1}{\sqrt{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}}} = -10log_{10}\left(1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}\right) \quad (3.4)$$

If we wish to satisfy our requirement of Ω_c at Ω_1 exactly and do better than our requirement at Ω_2 we use

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2n} = 10^{-0.1K_1} - 1 \quad (3.5)$$

while if we wish to satisfy our requirement at Ω_2 and exceed our requirement at Ω_1 we use

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2n} = 10^{-0.1K_2} - 1 \quad (3.6)$$

Dividing Eqn.(3.5) by (3.6) to cancel Ω_c we have

$$\left(\frac{\Omega_1}{\Omega_2}\right)^{2n} = \frac{10^{-0.1K_1} - 1}{10^{-0.1K_2} - 1} \quad (3.7)$$

A simple closed form answer for n is easily obtained from this expression and is given by

$$n = \left\lceil \frac{log_{10}[(10^{-0.1K_1} - 1)/(10^{-0.1K_2} - 1)]}{2log_{10}\left(\frac{\Omega_1}{\Omega_2}\right)} \right\rceil \qquad (3.8)$$

where $\lceil\cdot\rceil$ is the next larger integer.

Example 3.2:

a) Determine the order of a Butterworth filter that has a cutoff frequency of 1000 Hz and a gain of no more than -50 dB at 6000 Hz.

b) What is the actual gain in dB at 6000 Hz?

Solution:

a) The critical requirements are

$$\Omega_1 = \Omega_c = 2\pi(1000)\ rad/s \qquad K_1 = 20log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3dB$$

$$\Omega_2 = 2\pi(6000)\ rad/s \qquad K_2 \le -50\ dB$$

Substituting these requirements into Eqn.(3.8) gives

$$n = \left\lceil \frac{log_{10}\left[\left(10^{-0.1(-3)} - 1\right)/\left(10^{-0.1(-50)} - 1\right)\right]}{2log_{10}\left(\frac{2\pi(1000)}{2\pi(6000)}\right)} \right\rceil = \lceil 3.21 \rceil = 4$$

Therefore, we need a 4th order Butterworth filter.

b) We can use Eq. 3.4 to calculate the actual gain at 6000 Hz. The gain in decibels will be

$$K_{2(actual)} = 20\ log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{2\pi(6000)}{2\pi(1000)}\right)^{2(4)}}}\right) = -62.25\ dB$$

Example 3.3:

a) Determine the order of a Butterworth filter whose magnitude is 10 dB or better less than the passband magnitude at 500 Hz and at least 60 dB less than the passband magnitude at 5000 Hz.

b) Determine the cutoff frequency of the filter (in hertz).

c) What is the actual gain of the filter (in decibels) at 5000 Hz?

Solution:

a) The critical requirements are

$$\Omega_1 = 2\pi(500)rad/s \qquad K_1 = -10\ dB$$

$$\Omega_2 = 2\pi(5000)rad/s \qquad K_2 \le -60\ dB$$

$$n = \left\lceil \frac{log_{10}\left[\left(10^{-0.1(-10)} - 1\right)/\left(10^{-0.1(-60)} - 1\right)\right]}{2log_{10}\left(\frac{500}{5000}\right)} \right\rceil = \lceil 2.52 \rceil = 3$$

Therefore we need a 3rd order Butterworth filter to meet the specifications.

b) To do better at 500 Hz, we have to use Eq. 3.5, to determine the cutoff frequency.

$$\left(\frac{2\pi(500)}{\Omega_c}\right)^{2(3)} = 10^{-0.1(-10)} - 1$$

then, Ω_c= 2178.26 rad/s (f_c= 346.68 Hz)

c) The actual gain of the filter at 5000 Hz is

$$K_{2(actual)} = 20\ log_{10}\left(\frac{1}{\sqrt{1 + \left(\frac{5000}{346.68}\right)^{2(3)}}}\right) = -69.54\ dB.$$

3.3 Analog-to-Analog Transformations

If we replace s of $H(s)$, the system function for a normalized low-pass filter, by s/Ω_u, we get a new transfer function $H'(s)$, given by

$$H'(s) = H(s)\big|_{s\to s/\Omega_u} = H(s/\Omega_u)$$

If we evaluate the magnitude of the transfer function $H'(s)$ at $s = j\Omega$ to get the frequency response we have

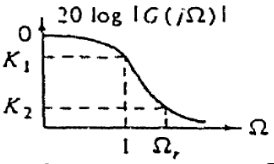
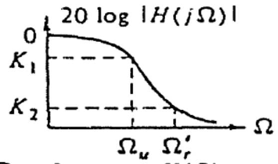
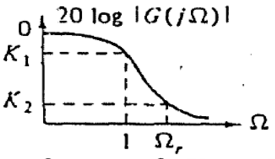
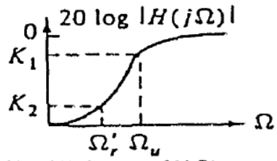
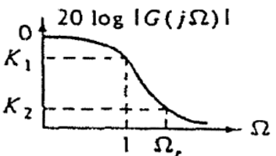
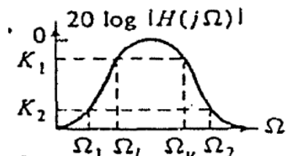
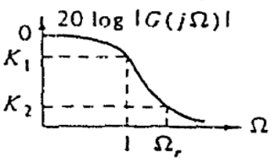
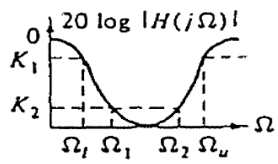
$$|H'(j\Omega)| = |H(j\Omega/\Omega_u)|$$

At the value of $\Omega = \Omega_u$ we have

$$|H'(j\Omega_u)| = |H(j\Omega_u/\Omega_u)| = |H(j1)|$$

That is, the frequency response for the new transfer function evaluated at $\Omega = \Omega_u$ is equal to the value of the normalized transfer function at $\Omega = 1$. <u>In a sense we have moved the cutoff frequency from 1 rad/sec to Ω_u rad/sec and thus have a scaling of the frequency axis</u>. Similar transformations can be defined for taking low-pass transfer functions to high-pass, bandpass and bandstop transfer functions. Table (3.2) gives these transformations.

TABLE 3.2 ANALOG-TO-ANALOG TRANSFORMATION

Prototype response	Transformed filter response	Design equations
 Low-pass $G(S)$	 $S \to S/\Omega_u$ Low-pass $H(S)$	Forward: $\Omega_r' = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega_r'/\Omega_u$
 Low-pass $G(S)$	 $S \to \Omega_u/S$ High-pass $H(S)$	Forward: $\Omega_r' = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega_r'$
 Low-pass $G(S)$	 $S \to \dfrac{S^2 + \Omega_l\Omega_u}{S(\Omega_u - \Omega_l)}$ Bandpass $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{av}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{av}^2 + \Omega_l\Omega_u)^{1/2} + \Omega_{av}\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
 Low-pass $G(S)$	 $S \to \dfrac{S(\Omega_u - \Omega_l)}{S^2 + \Omega_l\Omega_u}$ Bandstop $H(S)$	Forward: $\Omega_{av} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{av}/\Omega_r$ $\Omega_2 = [(\Omega_{av}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{av}/\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

<u>Example 3.4:</u> Design an analog Butterworth filter that has a –2 dB or better cutoff frequency of 20 rad/sec and at least 10 dB of attenuation at 30 rad/sec.

Solution. The critical requirements are

$\Omega_1 = 20, \qquad K_1 = -2 \quad , \qquad \Omega_2 = 30, \qquad K_2 = -10$

$$n = \left\lceil \frac{\log_{10}[(10^{0.2} - 1)/(10^{1} - 1)]}{2\log_{10}(20/30)} \right\rceil = \lceil 3.3709 \rceil = 4$$

Using this value of *n* to exactly satisfy the - 2 dB requirement gives

$$\Omega_c = 20/(10^{0.2} - 1)^{1/8} = 21.3868$$

The normalized low-pass Butterworth filter for *n* = 4, can be found from Table (3.1) as

$$H_4(s) = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

Applying a low-pass to low-pass transformation, $s \rightarrow s/\Omega_c$, with $\Omega_c = 21.3868$ gives the desired transfer function as follows:

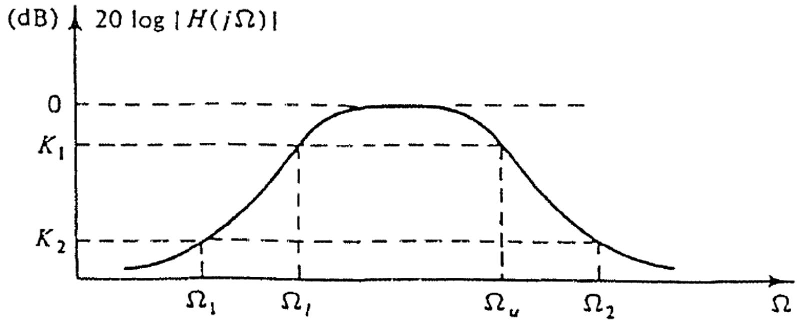
$$H(s) = H_4(s)|_{s\rightarrow s/21.3868} = \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.76536\left(\frac{s}{21.3868}\right) + 1\right]} \times \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 1.84776\left(\frac{s}{21.3868}\right) + 1\right]}$$

$$= \frac{2.09210 \times 10^5}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

3.4 Design of Bandpass Butterworth Filters

The procedures for the design of a bandpass filter $H_{BP}(s)$, to satisfy the given set of specifications is composed of two steps.

1. Design a low-pass filter $H_{LP}(s)$ with Ω_r,
2. Apply the low-pass to bandpass transformation using the desired Ω_u and Ω_l.



Example 3.5: Design an analog bandpass filter with the following characteristics:

(a) -3.0103 dB upper and lower cutoff frequency of 20 kHz and 50 Hz respectively.

(b) A stopband attenuation of at least 20 dB at 20 Hz and 45 kHz.

Solution: From the specifications above we can identify the following critical frequencies:

$\Omega_1 = 2\pi(20) = 125.663$ rad/sec

$\Omega_2 = 2\pi\,(45) \times 10^3 = 2.82743 \times 10^5$ rad/sec

$\Omega_u = 2\pi\,(20) \times 10^3 = 1.25663 \times 10^5$ rad/ sec

$\Omega_l = 2\pi\,(50) = 314.159$ rad/sec

Also the low-pass prototype must satisfy

$$0 \geqq 20\log|H_{LP}(j1)| \geqq -3.0103 \text{ dB}$$

$$20\log|H_{LP}(j\Omega_r)| \leqq -20 \text{ dB}$$

From Table (3.2)

$$A = 2.5053$$
$$B = 2.2545$$

Since,

$$\Omega_r = \min\{|A|, |B|\}$$