Some Topological properties of Julia sets of maps Of the form $\left(\lambda z - \lambda z^2\right)$

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Abstract

In this work, we study the topological properties of Julia sets of the quadratic polynomial maps of the form $\left(\lambda z - \lambda z^2\right)$ where λ is a non-zero complex constant .

Introduction

Complex dynamics is the study of iteration of maps which map the complex plane into itself . In general , their dynamics are quite complicated and hard to explain , but for some classes of maps , many interesting results can be proved . For example , one often studies the Julia sets of polynomial maps . The Julia set of the quadratic map of the form ($z^2 + c$) was studied extensively . From a dynamical systems point of view , all of the interesting behavior of a complex analytic map occurs on its Julia set , it is this set that contains the interesting topology [4] .The idea behind the Julia sets is to study whether the absolute value of a point in the complex plane converges towards infinity or not , when it is iterated under a map . All the points that do not go toward infinity , when iterated , are in the Julia set .

1 - Preliminary Definition

Let C be the complex set or complex plane,The complex plane together with the point at infinity , denoted by ∞ , is called the extended complex plane , it is topologically equivalent to the Riemann sphere. We put $C_\infty = C \cup \{\infty\}$ .The metric space of the complex plane is the usual metric , while the metric space of the Riemann sphere is the chordal metric .we use the symbol f^n to denote n-th iteration for $n \in N$. f : C $\to$ C is smooth , if f is a C^r - diffeomorphism if f is a C^r - homeomorphism such that f^{-1} is also C^r

. A point $x \in X$ is called a fixed point if $f(x) = x$. It is a periodic with period n if $f^n(x) = x$, but $f^m(x) \neq x$ for $m < n$.. Let x be a periodic point of period n for f . The point x is hyperbolic if $\left|(f^n)'(x)\right| \neq 1$, x is attracting periodic point if $\left|(f^n)'(x)\right| < 1$ and x is repelling periodic point if $\left|(f^n)'(x)\right| > 1$.

Remark (1-1)

The fixed points of $Q_\lambda(z) = \lambda z - \lambda z^2$ are $z = 0$ or $z = \frac{\lambda - 1}{\lambda}$. If if $z = 0$ then $\left|Q'_\lambda(0)\right| = |\lambda|$. If $|\lambda| < 1$, then $z = 0$ is attracting fixed point . If $|\lambda| > 1$, then $z = 0$ is repelling fixed point .If $z = \frac{\lambda - 1}{\lambda}$ then $\left|Q'_\lambda\left(\frac{\lambda - 1}{\lambda}\right)\right| = |2 - \lambda|$. If 3 < $|\lambda|$ or $|\lambda|$ < 1, then $z = \frac{|\lambda| - 1}{|\lambda|}$ is repelling fixed point . If 1 < $|\lambda|$ < 3 , then $z = \frac{|\lambda| - 1}{|\lambda|}$ is attracting fixed point . The critical point for Q_λ is 0.5 .

A is said to be completely invariant under f if $f(A) = A = f^{-1}(A)$.In [4] proved that $J(f)$ is completely invariant

There are many definition of Julia sets:

Definition (1-2) [2]

Suppose $f : C \to C$ is an analytic map . The Julia set is the closure of all repelling periodic points of f . That is

$J(f)$ = closure { all repelling periodic points of f } .

Definition (1-3) [1]

The family $\{f_n\}$ is said to be normal on U if every sequence of the f_n`s has a subsequence which either

1.converges uniformly on compact subsets of U ,or

2. converges uniformly to ∞ on U

Now, we will give the definition of the Fatou set and Julia set :

Definition (1-4) [2]

Let $f : C \to C$ be a map . The Fatou set (stable set) $F(f)$ is the set of points $z \in C$ such that the family of iterates $\{f^n\}$ is normal family in some neighborhood of z .The Julia set $J(f)$ is the complement of the Fatou set , that is $J(f)=\{\ z \in C$: the family $\{f^n\}_{n\geq 0}$ is not normal at $z\ \}$.

That is $J(f) \equiv C \setminus F(f)$.

Also the previous definition can satisfy on the space C_∞ .

Definition (1-5) [2]

Let $f : C_\infty \to C_\infty$ be a polynomial of degree $n \geq 2$. Let $K(f)$ denote the set of points in C whose orbits do not converge to the point at infinity . That is $K(f)=\{\ z \in C : \{|f^n(z)|\}_{n=0}^{\infty}$ is bounded $\}$. This set is called filled Julia set .

Definition (1-6) [2]

Let $f : C_\infty \to C_\infty$ be a map . The escape set $A(\infty)$ of f is all those points that escape to infinity , that is $A(\infty) = \{\ z : f^n(z) \to \infty$ as $n \to \infty\}$.

We can say that $A(\infty)$ is the basin of attraction of ∞ . Now we can state another definition for Julia set .

Definition (1-7) [2]

The Julia set is the boundary of the filled Julia set , that is $J(f)=\partial K(f)$. The complement of the basin of attraction of ∞ is the filled Julia set of f . That is $C_\infty \setminus A(\infty) = K(f)$.

Theorem (1-8)

Let $f : C \to C$ be a polynomial of degree $d \geq 2$. Then the following statements are equivalent .

1. $J(f)$ is the closure of repelling periodic points .
2. $J(f)$ is the complement of the Fatou set

3. $J(f)$ is the boundary of the filled Julia set .

Proof :

$1 \Leftrightarrow 2$ see [1]

$2 \Leftrightarrow 3$ From [2] , $J(f) = \partial A(\infty)$ .Since $J(f) = \partial K(f)$. ■

2.Julia Sets Properties

In this section , we will study the topological properties of Julia sets

Theorem (2-1) [3] (Bottcher theorem)

Let $f : C_\infty \to C_\infty$ be a rational map . If z_0 is super attracting fixed point , $f(z) = z_0 + a(z - z_0)^n + \ldots$, $n \geq 2$ and $a \neq 0$, then f is conjugate to φ o f o $\varphi^{-1} : w \to w^n$ in some neighborhood of z_0 .

The proof can be found in [3] .

Definition (2-2) [3]

Let $f : C_\infty \to C_\infty$ be a rational map has a super attracting fixed point at 0 . Let Ω be the basin of attraction of this fixed point . Define $G : \Omega - \{0\} \to R$ by $G(z) = \log|\varphi(z)|$. If $G(z)$ < 0 . Then the map is called the Green's maps of f .

Let $f : C_\infty \to C_\infty$ be a rational map . Define a critical value to be the image of a critical point. And, let a branch of the inverse map $f^{-1}(w)$ be the bijection between a neighborhood of w and a neighborhood of z where $f(z) = w$, w not a critical value of f .

Theorem (2-3)

The Julia set $J(Q_\lambda)$, where $Q_\lambda(z) = \lambda z - \lambda z^2$, is connected if and only if there is no finite critical point of Q_λ in the basin of attraction $A_\lambda(\infty)$.

Proof :

Let r(z)=1/z,then $F_\lambda(z) = r$ o Q_λ o $r(z) = r$ o $Q_\lambda\left(\frac{1}{z}\right) = r\left(\frac{\lambda}{z} - \frac{\lambda}{z^2}\right) = \frac{z^2}{\lambda z - \lambda}$. Thus $F_\lambda(0) = 0$. Therefore ∞ is super attracting fixed point of Q_λ and $Q_\lambda(\infty) = \infty$, also $Q'_\lambda(\infty) = 0$. Now in a neighborhood of ∞ and by theorem (2-1) there exists a conformal map φ such that $\varphi(Q_\lambda(z)) = \varphi(z)^2$ . . . (*), where $\varphi(z) = z + O(1)$ that is the following diagram commutes :

$$\begin{array}{ccc} & Q_\lambda(z) & \\ U_\infty & \longrightarrow & U_\infty \\ \varphi \downarrow & z^2 & \downarrow \\ U_\infty & \longrightarrow & U_\infty \end{array}$$

Such that $\varphi\, Q_\lambda\, \varphi^{-1} = g$ and $g(z) = z^2$. Next since $\log|\varphi(z)|$ has a logarithmic pole at ∞ , $\log|\varphi(z)| = \log|z + O(1)| \cong \log|z| = \log\sqrt{x^2 + y^2}$,

$\frac{\partial^2 \log|\varphi(z)|}{\partial x^2} + \frac{\partial^2 \log|\varphi(z)|}{\partial y^2} = 0$ .Thus $\log|\varphi(z)|$ is positive and harmonic .Since $J(Q_\lambda) = \partial A_\lambda(\infty)$ and if $|z| = 1$, then $\log|\varphi(z)| \to 0$ as $z \to \partial A_\lambda(\infty)$ .By definition (2-2) , thus $\log|\varphi(z)| = G(z)$ for $A_\lambda(\infty)$ .Thus taking the logarithm of the modulus of (*) for

$G(z)$, $\log|\varphi(Q_\lambda(z))| = 2\log|\varphi(z)|$, we have

$$G(Q_\lambda(z)) = 2G(z) \qquad . . . (**) .$$

Now a component of the Fatou set map onto another component of the Fatou set since otherwise a boundary point (in element of the julia set) map to a point in the interior of a component of the Fatou set . This is a contradiction because by[4], $J(Q_\lambda)$ is completely invariant . Next if a bounded component of $A_\lambda(\infty)$ exists , some iterates of Q_λ maps onto the component of $A_\lambda(\infty)$ which contains ∞ . This means that for some z in the bounded component and integer n , $Q_\lambda^n(z) = \infty$ .This is contradiction because the iterates of a polynomial are polynomials do not have poles . Thus $A_\lambda(\infty)$ is connected . Define a level curve of $G(z)$ as $\Lambda_a = \{z : G(z) = a\}$, where $a \in R$,since for $z \in \Lambda_a$, $G(Q_\lambda(z)) = 2G(z) = 2a$, then $Q_\lambda(z)$ takes the level curve Λ_a to the level curve Λ_{2a} . So $Q_\lambda(z) \in \Lambda_{2a}$. Define the exterior of level curve Λ_a to be the set $E_a = \{ z : G(z) > a \} = \{ z : |\varphi(z)| > e^a \}$.

Then $Q_\lambda(z)$ maps E_a two –to-one to E_{2a} which is a subset of E_a . To extend $\varphi(z)$, first consider a neighborhood $U = E_r$ of ∞ on which the theorem (2-1) holds . Then on $E_{\frac{r}{2}}$ we can define $\varphi(z) = \sqrt{\varphi(Q_\lambda(z))}$ (since $z \in E_{\frac{r}{2}}$, $Q_\lambda(z) \in E_r$) so the right hand side of the equation is defined . Continue in this way defining $\varphi(z)$ on $E_{\frac{r}{2^n}} = \{ z : |\varphi(z)| >$ exp($\frac{r}{2^n}$)} as long as there are no critical point in the extended region . At a critical point a single –valued analytic map can not be defined . So as $n \to \infty$, φ is defined on $\bigcup_{n=1}^{\infty} E_{\frac{r}{2^n}} = \{ z : |\varphi(z)| >$ exp($\frac{r}{2^n}$)} , that is

$\bigcup_{n=1}^{\infty} E_{\frac{r}{2^n}} = \{ z : |\varphi(z)| > 1\} = \{ z : G(z) > 0\} = A_\lambda(\infty)$.

$(\Leftarrow)$ Recall that $A_\lambda(\infty)$ is connected . Now φ is homeomorphism which maps $A_\lambda(\infty)$ conformally to the exterior of the unit disk . Since simple connectivity is preserved by homeomorphism and exterior of the unit disk on the Riemann sphere is simply connected , $A_\lambda(\infty)$ must be simply connected . Thus it follows that $\partial A_\lambda(\infty) = J(Q_\lambda)$ is connected .

$(\Rightarrow)$ Assume that there exist $z_0 \in \partial A_\lambda(\infty)$, z_0 is a finite critical of $Q_\lambda(z)$. Let $G(z_0) = r_0$ and consider Λ_{r_0} . Differentiating (**) at z_0 yields $\left(\frac{\partial}{\partial z} G(Q_\lambda(z_0))\right) . Q'_\lambda(z_0) = 2 \frac{\partial}{\partial z} G(z_0)$, since z_0 is a critical point of Q_λ , $Q'_\lambda(z_0) = 0$, thus $\left(\frac{\partial}{\partial z} G(Q_\lambda(z_0))\right) . Q'_\lambda(z_0) = 2 \frac{\partial}{\partial z} G(z_0) = 0 = 2 \frac{\partial}{\partial z} G(z_0)$, thus $\frac{\partial}{\partial z} G(z_0) = 0$ .So z_0 is a critical point of $G(z)$. Thus the level curve Λ_{r_0} consists of at least two simple closed curves that meet at the critical point z_0 . Within each of these simple curves there exist points in the Julia set. If not , $G(z)$ is harmonic and positive on a non-empty region V within one of the simple curves and the maximum principle applied to $G(z)$ and $-G(z_0)$ gives $G(z) \le r_0$ and $-G(z_0) \le -r_0$ for all $z \in V$. So $G(z) \equiv r_0$ on V . Let f be the analytic map with real part equal to $G(z)$. Then by the uniqueness theorem , $G(z) \equiv r_0$ on $A_\lambda(\infty)$. This contradicts that $A_\lambda(\infty) \to \infty$. Thus $J(Q_\lambda)$ is disconnected . Therefore there is no finite critical point of Q_λ in $A_\lambda(\infty)$. ■

Proposition (2-4)

If Q_λ has a critical point in $A_\lambda(\infty)$, then $J(Q_\lambda)$ has uncountably many components.

Proof :

Let z_0 be a critical point for Q_λ .Let w be an element of the backward orbit of z_0 , that is $Q_\lambda^n(w) = z_0$ for some n . Then by theorem (2-3),that G defined on this theorem , $G(Q_\lambda^n(w)) = 2^n G(w)$, or $G(w) = 2^{-n} G(Q_\lambda^n(w))$. Thus $G(w) = 2^n G(z_0)$. Differentiate both sides to get $\frac{\partial}{\partial z} G(w) = 2^n \frac{\partial}{\partial z} G(z_0) = 0$,so that w is a critical point of $G(z)$ .Thus any level curve consists of at least two simple closed curves that meet at the critical point w . Since the choice of w was arbitrary , the level curves split in each of the w ,so follow the splitting by assigning 0 to the left branch and 1 to the right branch .Since there are uncountably many sequences of 0`s and 1`s there are uncountably many components of $J(Q_\lambda)$. ■

Definition (2-5) [1]

A set is totally disconnected If it contains no intervals .

Theorem (2-6)

Let $Q_\lambda(z) = \lambda z - \lambda z^2$. If $Q_\lambda^n(0.5) \to \infty$, then $J(Q_\lambda)$ is totally disconnected .

Proof :

Since $\infty \in F(Q_\lambda)$ and $F(Q_\lambda)$ is open , there exists a neighborhood D_∞ of ∞ such that $\overline{D_\infty} \subset F(Q_\lambda)$. And since ∞ is an attracting fixed point of Q_λ , $Q_\lambda(\overline{D_\infty}) \subset D_\infty$.

Let $D = C_\infty \setminus \overline{D_\infty}$. Then D is an open set and $J(Q_\lambda) \subset D$.

Now, since $Q_\lambda^n(0.5) \to \infty$ by assumption , choose k large enough so that Q_λ^k maps 0.5 to D_∞ . Thus for $n \geq k$, there is no critical value of Q_λ^n in $\overline{D}$, and all the branches of the inverse map Q_λ^{-n} are defined and map $\overline{D}$ in D . (Else there exists $z \in \overline{D}$ such that $w = Q_\lambda^{-n}(z) \in (C_\infty \setminus D) = D_\infty$. Now $Q_\lambda^n(w) = z \in \overline{D}$, but $Q_\lambda^n : \overline{D_\infty} \to D_\infty$ implies that $z \in D_\infty$. This contradicts the choice of $z \in \overline{D}$) .Let $z_0 \in J(Q_\lambda)$, then $Q_\lambda^n(z_0) \in J(Q_\lambda)$ since the Julia set is completely invariant under Q_λ from [4] . Define f_n to be the branch of the inverse map Q_λ^{-n} which maps $Q_\lambda^n(z_0)$ to z_0 . That is , $f_n(Q_\lambda^n(z_0)) = z_0$. Since f_n maps $\overline{D}$ into D, $\{f_n\}$ are uniformly bounded on $\overline{D}$ .Note that by modifying the integer k above , $\{f_n\}$ is uniformly bounded on a neighborhood of $\overline{D}$. Thus $\{f_n\}$ is normal on $\overline{D}$.Now for all $z \in D \cap A_\lambda(\infty)$, $f_n(z)$ accumulates on $J(Q_\lambda)$ since $f_n(z) \to \partial A_\lambda(\infty)$ (except for $z = \infty$) . Then let f be the limit of some subsequence $\{f^{n_m}\}$ of $\{f_n\}$ .Now f maps $D \cap A_\lambda(\infty)$ into $J(Q_\lambda)$ since $f^n(z) \to w \in J(Q_\lambda)$ and $f^{n_m} \to f(z)$ implies $f(z) = w \in J(Q_\lambda)$ by the uniqueness of limits . Then by open map see[1] f is constant since $J(Q_\lambda) = \partial A_\lambda(\infty)$ the boundary of an open set has empty interior . (If z in the interior of the boundary of an open set U then there exist a neighborhood of z contained entirely in ∂U .But for any ε>0 , there exists $z_1 \in D(z,\varepsilon) \cap U$, by the definition of the boundary set . This contradicts that U is open) . Now , diam $\{f_n(D)\} \to 0$. (Suppose not . Then there exists ε>0 and $\{f_{n_m}\}$ such that diam $\{f_{n_m}(D)\} \geq \varepsilon$. $\{f_{n_m}\}$ is normal so there exists $\{f_{n_{m_j}}\}$, a subsequence , and f a limit map , such that $f_{n_{m_j}} \to f$ uniformly . By the argument in the previous paragraph , $f \equiv w_0$, a constant . Thus for a fixed branch , $f_{n_{m_j}}$, with $j \geq j_0$, $\left| f_{n_{m_j}}(z) - w_0 \right| < \dfrac{\varepsilon}{3}$ for all $z \in \overline{D}$, Then diam $\{f_{n_{m_j}}(D)\} < \dfrac{2\varepsilon}{3}$, contradiction

) . Then since f_n is continuous , $f_n(\overline{D}) \subseteq \overline{f_n(D)}$, and $f_n(\overline{D})$ has diameter tending to zero . Next , by invariance of Fatou set , $\partial D \subset F(Q_\lambda)$ implies that $f_n(\partial D) \subset F(Q_\lambda)$ and it is disjoint from $J(Q_\lambda)$. Now recall that for $z \in J(Q_\lambda)$, f_n was chosen so that $f_n(Q_\lambda^n(z_0)) = z_0$ and $f_n(Q_\lambda^n(z_0)) \subset f_n(D)$ since $Q_\lambda^n(z_0) \in$ $J(Q_\lambda) \subset D$. Also , $f_n(D) \subset f_n(\overline{D})$, so $z_0 \in$ $f_n(\overline{D})$ for all n . Now diam $\{f_n(\overline{D})\} \to 0$ implies that $\{z_0\}$ must be a connected component of . To see this recall that $\overline{D}$ consists of elements of $J(Q_\lambda)$ and the boundary which is in $F(Q_\lambda)$. For any ε >0 ,choose k such that diam $\{f_k(\overline{D})\} < \varepsilon$. Within this disc $f_k(\overline{D})$, the boundary is mapped to a curve that winds around elements of the Julia set in the interior . Thus only points in the Julia set are within 2ε of each other will be elements of a connected component of $J(Q_\lambda)$. But since ε can be chosen arbitrarily small , eventually all points of the Julia set will be separated by the Fatou set , $\partial f_k(D)$ for large enough k .By definition (2-6) , $J(Q_\lambda)$ is totally disconnected . ■

Corollary (2-7)

Let z be the critical point of $Q_\lambda(z) = \lambda z - \lambda z^2$. If $Q_\lambda^n(z) \to \infty$, then $J(Q_\lambda)$ is totally disconnected . Otherwise, $\{Q_\lambda^n(z)\}$ is bounded , and $J(Q_\lambda)$ is connected .

Proof :

Since $z = 0.5$ is the critical point of $Q_\lambda(z)$ .If $Q_\lambda^n(0.5)$ is bounded then $0.5 \in A_\lambda(\infty)$ and $J(Q_\lambda)$ is connected from theorem (2-3) .Next , if $Q_\lambda^n(0.5) \to \infty$ then by theorem (2-6) $J(Q_\lambda)$ is totally disconnected . ■

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