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Domination in Isosceles Triangular Chessboard

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Abstract

In this article, we are interested in the domination problem in isosceles triangular chessboard. In our study we take in account one type of piece of rooks, bishops and kings, and two different types of pieces together, kings with rooks, kings with bishops and rooks with bishops. The chessboard in this work is isosceles triangular with square cells.

Also in most cases we determine the possible number of different methods of domination (total solution).

Keywords: domination, Isosceles triangle chessboard, Kings, Bishops and Rooks.

1. Introduction

One of the classical chessboard problems is placing minimum number of one type of pieces such that all unoccupied positions are under attack. This problem is called the domination number problem of P. This number is denoted by $\gamma(\mathrm{P})$ and we denote the number of different methods for placing P pieces to obtain the domination number of P by $\mathrm{S}(\gamma(\mathrm{P}))$. Also we are interested in the domination number of two different types of pieces by fixing a number n_p of one type P of pieces and determine the domination number $\gamma(P^*, n_p)$ of another type $P^*$of pieces. Finally we compute $S(\gamma(P^*, n_p))$.

In n square chessboard (see [4], [5] and [6]) γ is studied for rook "R", bishop "B" and king "K" .They proved that $\gamma(\mathrm{R}) = n$, $\gamma(\mathrm{B}) = n$ and $\gamma(\mathrm{K}) = \left\lfloor\frac{n+2}{3}\right\rfloor^2$.

In [3], JoeMaio and William proved that $\gamma(\mathrm{R}) = \min\{\mathrm{m}, n\}$ for mxn Toroidal chessboards.

A saw-toothed chessboard, or STC for short, is a kind of chessboard whose boundary forms two staircases from left down to right without any hole inside it.

In [2] Hon-Chan Chena, Ting-Yem Hob, determined the minimum number of rooks that can dominate all squares of the STC.

Dietrich and Harborth [1] studied the triangular triangle board, i.e. the board in the shape of a triangle with triangular cells. They defined the chess pieces in particular the rook which attacks in straight lines from a side of the triangle to a side of the triangle, forming rhombuses, the bishop 1 which attacks from vertex to side, side to vertex etc. in straight lines, forming diamonds, and for the bishop 2: the triangular triangle board can be 2-coloured, whose cells are sharing an edge of different color, and the bishop 2 moves as bishop 1, but attacks only cells of the same color.

2. Chessboard of two equal sides length

In this work, we consider the isosceles triangular chessboard with square cells and three pieces, rooks, bishops and kings. They move or attack the pieces as usual.

We mean by the length of the two equal sides of the board the number of cells (squares) in each side. Let the length of each side is n, consequently the third side (base) is of length $2n - 1$.

In matrix form, let r_i denote the i^{th} row measured from above to down, i = 1, 2,…, n. If L_i denotes the length of row r_i, then the first row r_1 which contains one cell will have the length L_1= 1, the second row r_2 which contains 3 cells has the length L_2 = 3, and so on… In general the i^{th} row which contains $2i - 1$ cells has the length $L_i = 2i - 1$. Let c_j denote the j^{th} column which is numbered from the middle (the middle column has the greatest length of columns), j = 0, ±1, ±2, … , ±(i – 1). Let the middle column c_0 contain n cells, then each of the two columns c_1 and c_{-1} which lies to the right and to the left of c_0 respectively contains n-1cells. In general the j^{th} column contains n – |j| cells. We denote the cell (square) of row i and column j by $s_{i,j}$, $i = 1, 2, \dots n$, and j = 0, ±1, ±2, … , ±(i – 1) and we note that the number of squares in isosceles triangular chessboard of two equal sides of length n is n^2. We refer to the length of the two equal sides by the size of our chessboard. Figure 1 shows a chessboard of lengths 5, 5, 9.

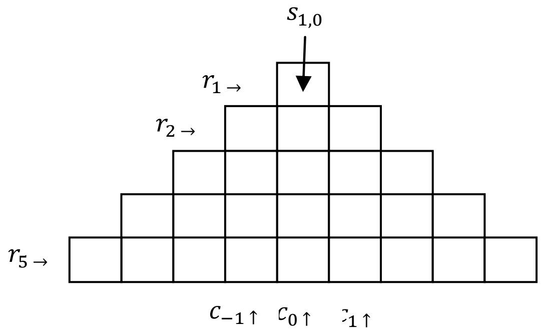


Figure 1

3. Domination of one piece

In this section we will compute the domination number $\gamma(P)$ of one type of piece and it is total solution $S(\gamma(P))$.

3.1. Domination of Rooks

Theorem 3.1.1.

(I) $\gamma(\mathrm{R}) = \mathrm{n} - \left\lfloor \frac{\mathrm{n}+1}{3} \right\rfloor$

(II) $\left(\mathrm{n} - \left\lfloor \frac{\mathrm{n}+1}{3} \right\rfloor\right)! \le \mathrm{S}(\gamma(\mathrm{R})) \le \left(2\left\lfloor \frac{\mathrm{n}+1}{3} \right\rfloor + 1\right)!$

Proof: (I) We note that by putting a piece in each cell of c_0, we can dominate the chessboard. To find $\gamma(\mathrm{R})$, we look for the largest number of rows beginning from r_1 which can be dominated without putting any piece in it. Suppose that this number is i_0, consequently the number of pieces dominating the empty (remaining) rows is n $- \mathrm{i}_0$ pieces . This means that $\gamma(\mathrm{R}) = n - \mathrm{i}_0$. And now if $n - \mathrm{i}_0 < \mathrm{L}_{\mathrm{i}_0}$, then we cannot dominate the row i_0, since the number of pieces is less than $\mathrm{L}_{\mathrm{i}_0}$. Hence $n - \mathrm{i}_0 \ge 2\mathrm{i}_0 - 1$, which gives $\mathrm{i}_0 = \left\lfloor \frac{\mathrm{n}+1}{3} \right\rfloor$, thus $\gamma(\mathrm{R}) = \mathrm{n} - \left\lfloor \frac{\mathrm{n}+1}{3} \right\rfloor$.

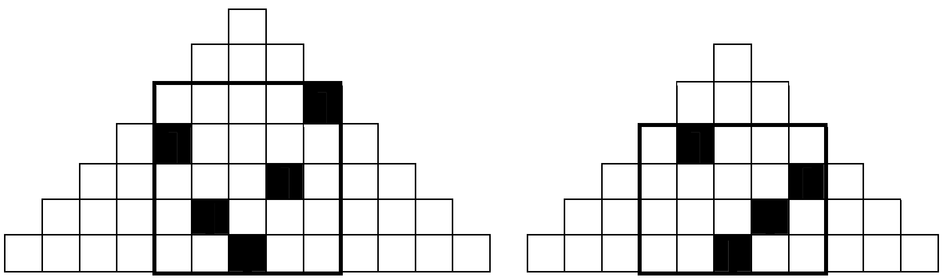
(II) We have two cases as follows:

(i) If $n - \left\lfloor \frac{n+1}{3} \right\rfloor = 2\left\lfloor \frac{n+1}{3} \right\rfloor + 1$, we distribute $\gamma(\mathrm{R})$ pieces on a square of "dimension" $n - \left\lfloor \frac{n+1}{3} \right\rfloor$ (see Figure 2 (a); $n = 7$). The number of methods to put one piece in the $\mathrm{r}_{\mathrm{i}_0+1}$ is $\mathrm{L}_{\mathrm{i}_{0+1}}$, the number of methods to put one piece in the next row $\mathrm{r}_{\mathrm{i}_0+2}$ is $\mathrm{L}_{\mathrm{i}_{0+1}} - 1$ and so on ... Finally the number of methods to put one piece in the last row r_n is one. Thus the total number of Dom(R) is $(\mathrm{L}_{\mathrm{i}_0+1})(\mathrm{L}_{\mathrm{i}_0})(\mathrm{L}_{\mathrm{i}_0-1}) \ldots (1) = (2\mathrm{i}_0 + 1)!$,

so $\mathrm{S}(\gamma(\mathrm{R})) = \left(n - \left\lfloor \frac{n+1}{3} \right\rfloor\right)!$.

(ii) If $n - \left\lfloor \frac{n+1}{3} \right\rfloor < 2\left\lfloor \frac{n+1}{3} \right\rfloor + 1$, then this is similar to (i), but we distribute $\gamma(\mathrm{R})$ pieces on a rectangular chessboard of dimensions $n - \left\lfloor \frac{n+1}{3} \right\rfloor$ and $2\left\lfloor \frac{n+1}{3} \right\rfloor + 1$ (see Figure 2(b), $n = 6$), such that one of the pieces must be put in the column c_0.

By using (i), we get $\left(n - \left\lfloor \frac{n+1}{3} \right\rfloor\right)! < \mathrm{S}(\gamma(\mathrm{R})) < \left(2\left\lfloor \frac{n+1}{3} \right\rfloor + 1\right)!$.



a b

Figure 2

3.2 Domination of Bishops

Theorem 3.2.1.

(I) $\gamma(B) = n$

(II) $S(\gamma(B)) = \begin{cases} n & if\ n\ is\ odd \\ n-1 & if\ n\ is\ even \end{cases}$

Proof:

It is clear that, if we put one piece in each cell of c_0 of an isosceles triangular chessboard, we get $\gamma(B)$, if we put any piece except the piece in $s_{n,0}$ in another place we will need another piece for dominating the chessboard. The piece in $s_{n,0}$ can move in $n-1$ cells when n is even, and in n cells when n is odd.

3.3. Domination of Kings

In this section, for the total solution S, we are interested in the distribution of pieces in row movement, since in column movement, the outcome is much more difficult. We denote the solution of domination of kings by $S^*(\gamma(K)$.

Theorem 3.3.1.

(I) $\gamma(K) = \begin{cases} \frac{\left\lceil\frac{2n-1}{3}\right\rceil}{4}\left(\left\lceil\frac{2n-1}{3}\right\rceil+2\right), if \left\lceil\frac{2n-1}{3}\right\rceil\ is\ even \\ \frac{\left(\left\lceil\frac{2n-1}{3}\right\rceil+1\right)^2}{4}, \quad if \left\lceil\frac{2n-1}{3}\right\rceil\ is\ odd \end{cases}$

(II) $S^*(\gamma(K)) = \begin{cases} \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-1}\left\lceil\frac{2n-(3+6w)}{3}\right\rceil & if\ n \equiv o(mod\ 3) \\ \prod_{w=1}^{\left\lceil\frac{n}{3}\right\rceil-1}(2w^2+w) & , if\ n \equiv 1(mod\ 3) \\ 1 \quad , & if\ n \equiv 2(mod\ 3) \end{cases}$

Proof:

(I) We use row r_{n-1} to dominate the cells of r_n, r_{n-1} and r_{n-2} . Since we need one piece to dominate every three successive cells and row r_n contains $2n-1$ cells, then we use $\left\lceil\frac{2n-1}{3}\right\rceil$ pieces to dominate the cells of r_n, r_{n-1} and r_{n-2}. We distribute these cells in r_{n-1} such that one piece for each corner: $s_{n-1,-(n-2)}$ and $s_{n-1,(n-2)}$, to guaranty domination of the two cells $s_{n,-(n-2)}$ and $s_{n,(n-2)}$ .By the same manner, we use r_{n-4} to dominate the cells of r_{n-3}, r_{n-4} and r_{n-5}. Thus we distribute $\left\lceil\frac{2n-7}{3}\right\rceil$ pieces in r_{n-4} to dominate the cells in the three rows. And so on…, we obtain

$$\gamma(K) = \sum_{w=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\left\lceil\frac{2n-(1+6w)}{3}\right\rceil = \begin{cases} \frac{\left\lceil\frac{2n-1}{3}\right\rceil}{4}\left(\left\lceil\frac{2n-1}{3}\right\rceil+2\right), if \left\lceil\frac{2n-1}{3}\right\rceil\ is\ even \\ \frac{\left(\left\lceil\frac{2n-1}{3}\right\rceil+1\right)^2}{4}, \quad if \left\lceil\frac{2n-1}{3}\right\rceil\ is\ odd \end{cases}$$

(II) We have three cases as follows:

(i) $n \equiv 0$ mod 3 , as in (I), we distribute the pieces in row r_{n-1} to dominate the last three rows r_n, r_{n-1} and r_{n-2} such that one piece is in each of the corners $s_{n-1,-(n-2)}$ and $s_{n-1,(n-2)}$. We distribute the king pieces from right to left such that between any two pieces there are two empty cells, and there is one empty cell between the last two pieces (see Figure 3; n = 9).

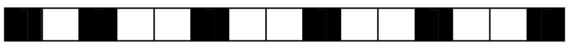


Figure 3

Now we study the possibility of the movement of the pieces from left to right (there is no possibility in this case for movement from right to left). Thus any one move of a piece gives one method of domination, so the total number of methods of domination is $\left\lceil\frac{2n-3}{3}\right\rceil$ (see Figure 4; $n=9$). Similarly we use row r_{n-4} to dominate the next above three rows r_{n-3}, r_{n-4} and r_{n-5}. The number of methods of domination is $\left\lceil\frac{2n-9}{3}\right\rceil$, and so on... Thus the general formula of the total methods of domination is given by $S^*(\gamma(K)) = \prod_{w=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\left\lceil\frac{2n-(3+6w)}{3}\right\rceil$.

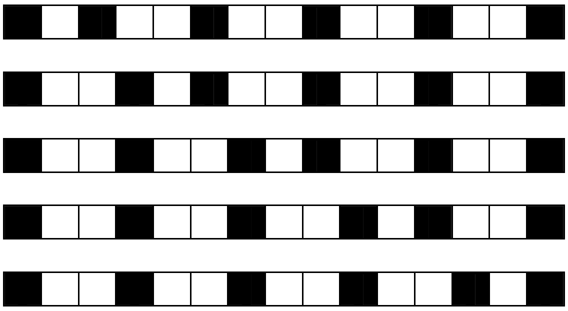


Figure 4

(ii) $n \equiv 1$ mod 3

Again to dominate the three rows r_n, r_{n-1} and r_{n-2}, we start distributing the pieces in row r_{n-1}. In this case the last two pieces will be in the cells $s_{n-1,-(n-2)}$ and $s_{n-1,-(n-3)}$ on the left as shown in Figure 5; n = 10.

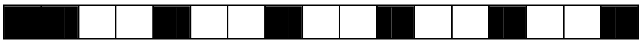


Figure 5

We start the movement of the pieces from left to right as in the following steps:
Step 1) As shown in (Figure 5; $n = 10$), if we fix the pieces in $s_{n-1,-(n-2)}$ and $s_{n-1,-(n-3)}$, then we have a unique method of domination in this step, since there exist two empty cells between any two other pieces.
Step 2) When we move the piece from $s_{n-1,-(n-3)}$ to $s_{n-1,-(n-4)}$ we obtain a new method of distribution. This move will allow us to move all the next pieces to the right direction, this means we obtain a method for the movement of any piece. Thus the number of methods in this step is $\left(\left\lfloor\frac{2n-1}{3}\right\rfloor - 1\right)$.
Step 3) The second move start form $s_{n-1,-(n-3)}$ to $s_{n-1,-(n-5)}$.Thus we have two pieces in the cells $s_{n-1,-(n-5)}$, $s_{n-1,-(n-6)}$ (see Figure 6; $n = 10$).

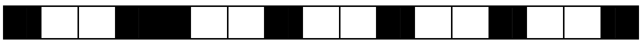


Figure 6

Again with the same manner (as in step 2), we have a number of methods which is equal to $\left\lfloor\frac{2n-1}{3}\right\rfloor - 2$, and so on. Thus the corresponding total solution of movement in row r_{n-1} is $\left\lfloor\frac{2n-1}{3}\right\rfloor + \sum_{i=0}^{\left\lfloor\frac{2n-1}{3}\right\rfloor-3}\left(\left\lfloor\frac{2n-1}{3}\right\rfloor - 1 - i\right)$.
Step 4) With the same manner we can determine the methods of distribution in any other rows that contain pieces. Finally for the row r_1 which contains one cell, we have one method only for domination. So this row has no influence on the product of the general formula, and consequently the number of rows which we deal with is $\left\lceil\frac{n}{3}\right\rceil - 1$ (since the number of rows that contains kings is $\left\lceil\frac{n}{3}\right\rceil$). Also we note that the rows which contain pieces have the same manner of distributions, since all their lengths are equal in mod 3.Thus the total number of methods of domination is given by

$$S^*(\gamma\ (\mathrm{K})) = \prod_{w=1}^{\left\lceil\frac{n}{3}\right\rceil-1}\left\{\left\lfloor\frac{2(1+3w)-1}{3}\right\rfloor + \sum_{i=0}^{\left\lfloor\frac{2(1+3w)-1}{3}\right\rfloor-3}\left(\left\lfloor\frac{2(1+3w)-1}{3}\right\rfloor - 1 - i\right)\right\} = \prod_{w=1}^{\left\lceil\frac{n}{3}\right\rceil-1}(2w^2 + w)$$

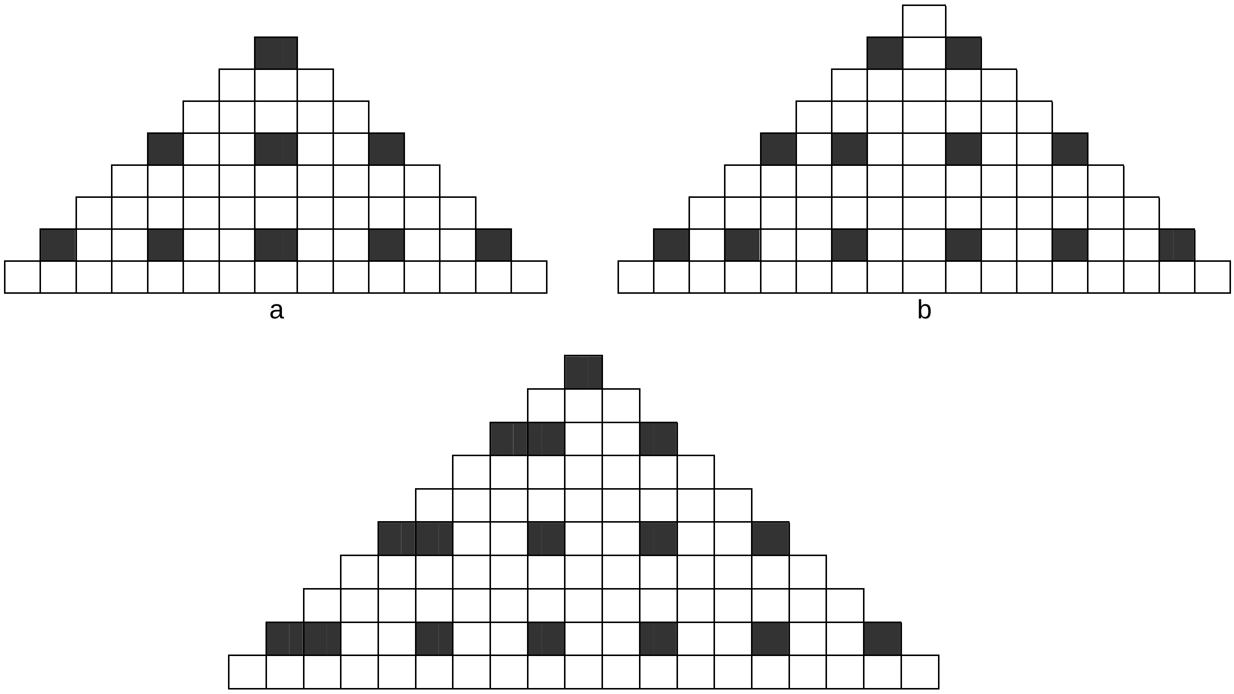
(iii) $n \equiv 2 \ \ mod\ 3$

Again, we start with row r_{n-1} to distribute the king pieces .We fix two pieces in $s_{n-1,-(n-2)}$ and $s_{n-1,(n-2)}$. Naturally we must distribute the other pieces such that there are two empty cells between any two pieces of them. It is clear that there exists a unique method to distribute the minimum number of kings, since we cannot move any piece to any cell. So $S\ (\gamma\ (K)) = 1$.

The following example is illustrating the above theorem for some different values of n.

Example 3.3.2. Let n = 8, 9, 10, determine $\gamma(K)$ and S^* (γ (K)) for all n.

1) If $n =$ 8, then $\gamma(K) = 9$ and S^* (γ (K)) = 1, (see Figure 7 (a)) .
2) If $n =$ 9, then $\gamma(K) = 12$ and S^* (γ (K)) = 15, (see Figure 7 (b)) .
3) If $n =$ 10, then $\gamma(K) = 16$ and S^* (γ (K)) = 630, (see Figure 7 (c)) .



a b

c

Figure 7

We need the following corollary concerning the equilateral triangular chessboard of length n for the next section dealing with two pieces.

Corollary 3.3.3. For an equilateral triangular chessboard of length n:

(I) $\gamma(K)=\sum_{w=0}^{\left\lfloor\frac{n-1}{3}\right\rfloor}\left\lceil\frac{n-3w}{3}\right\rceil=\binom{\left\lceil\frac{n}{3}\right\rceil+1}{\left\lceil\frac{n}{3}\right\rceil-1}\ \forall\, n\geq 2$

(II) $$S^{*}(\gamma\,(K\,))=\begin{Bmatrix} 1 & , & if\ n\equiv 0(mod\ 3\,) \\ \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-2}\frac{1}{2}\left(\left\lfloor\frac{n-3w}{3}\right\rfloor^{2}+3\left\lfloor\frac{n-3w}{3}\right\rfloor\right) & , & if\ n\equiv 1(mod\ 3\,) \\ \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-1}\left(\left\lceil\frac{n-3w}{3}\right\rceil\right) & , & if\ \ n\equiv 2(mod\ 3\,) \end{Bmatrix}$$

Proof:

(I) Figure 8 gives an example of the equilateral triangular chessboard of $n=8$. With the same methods of Theorem 3.3.1., the assertion is clear.

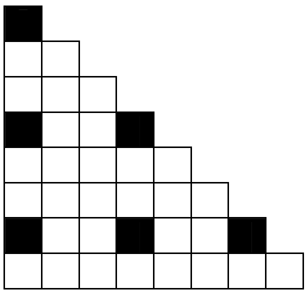


Figure 8

(II) We have three cases as follows:

(i) $n\equiv 0\ \ mod\ 3$

To distribute king pieces from right to left, we start with row r_{n-1} and fix one piece in cell $s_{n-1,n-2}$. Naturally we must distribute the other pieces such that there are two empty cells between any two of them. We are noting that the last king exists in $s_{n-1,1}$ and the cell $s_{n-1,0}$ is empty as shown in Figure 9; $n=12$.

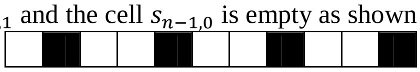


Figure 9

Therefore we cannot move any piece in any direction, so S(γ (K))=1.

(ii) $n\equiv 1\ \ mod\ 3$

To dominate the three rows r_n, r_{n-1} and r_{n-2}, we start distributing the pieces in row r_{n-1} as in (i). In this case there is one empty cell between the last two pieces in cells $s_{n-1,0}$ and $s_{n-1,2}$ as shown in Figure 10; n = 10.

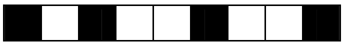


Figure 10

If we fix the king piece of king in cell $s_{n-1,0}$ and move the other pieces to the right direction then the number of methods is $\left\lfloor\frac{n-3w}{3}\right\rfloor$, $w = 0,1,\dots,\left\lceil\frac{n}{3}\right\rceil - 2$. Now we move the king from $s_{n-1,0}$ to $s_{n-1,1}$, we get the last two kings are in adjacent cells shown in Figure 11; $n = 10$.

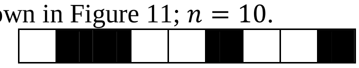


Figure 11

Similar to part (ii) of Theorem 3.3.1 (III), The total solution is given

$S^*(\gamma(K))=\prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-2}\frac{1}{2}\left(\left\lfloor\frac{n-3w}{3}\right\rfloor^2+3\left\lfloor\frac{n-3w}{3}\right\rfloor\right)$

(iii) $n \equiv 2 \mod 3$

Again we start with row r_{n-1} to distribute the king pieces from right to left. If we fix one piece in the cell $s_{n-1,n-2}$, then there exist two empty squares between any two pieces of kings as shown in Figure 12; $n = 11$.

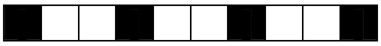


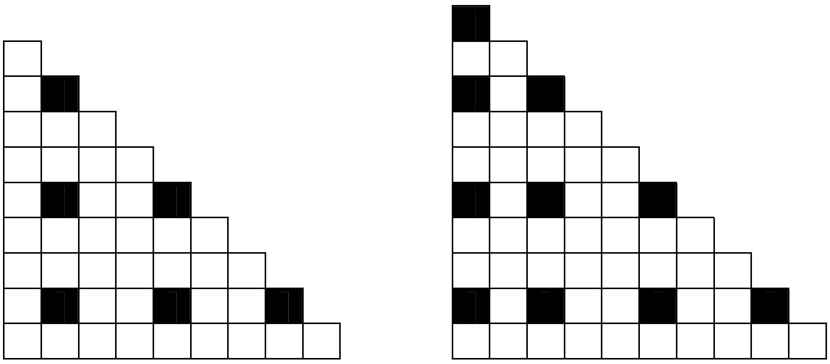
Figure 12

So we cannot move any pieces to right cell. Now it is possible to move any piece to the left cell, so every movement takes one method of solution, that means the number of methods in this row is $\left\lceil\frac{n}{3}\right\rceil$, Thus the total number of methods of domination is given by $S^*(\gamma(K)) = \prod_{w=0}^{\left\lceil\frac{n}{3}\right\rceil-1}\left(\left\lceil\frac{n-3w}{3}\right\rceil\right)$.

The following example is illustrating the above Corollary for some different values of n.

Example 3.3.4. Let $n = 8,9,10$, we have $\gamma(K)$ and S(γ (K)) for all n as the follows:

1) If $n =$ 8, then $\gamma(K) = 6$ and $S^*(\gamma(K))$ = 6, (see Figure 8).
2) If $n =$ 9, then $\gamma(K) = 6$ and $S^*(\gamma(K))$ = 1, (see Figure 13 (a)).
3) If $n =$ 10, then $\gamma(K) = 10$ and $S^*(\gamma(K))$ = 90, (see Figure 13 (b)).



a b

Figure 13

4. Domination of Two pieces

In this section, we shall fix a number of one type P and determine the domination number of the other type of piece P^*. By n_P we mean the fixed number of pieces P and by N_P the number of the cells which are attacked by the piece P and the cell of that piece.

4.1 Domination of kings with a fixed number of rooks

We denote the domination number of kings with a fixed number of rooks n_r by $\gamma(K, n_r)$.

Theorem 4.1.1.

(I) $$\gamma(K\,,n_r)=\left\{\begin{matrix} 2\dbinom{\left\lceil\frac{n-3k-2}{3}\right\rceil+1}{\left\lceil\frac{n-3k-2}{3}\right\rceil-1} & ,\text{if } n_r \text{ is odd} \\ \dbinom{\left\lceil\frac{n-3k}{3}\right\rceil+1}{\left\lceil\frac{n-3k}{3}\right\rceil-1}+\dbinom{\left\lceil\frac{n-3k-1}{3}\right\rceil+1}{\left\lceil\frac{n-3k-1}{3}\right\rceil-1} & ,\text{ if } n_r \text{ is even}\end{matrix}\right\}$$

(II) If

$$\mu=\left\{\begin{matrix} 1 & , & if\ n-3k-2\equiv o(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k-2}{3}\right\rceil-2}\frac{1}{2}\left(\left\lceil\frac{n-3k-2-3w}{3}\right\rceil^2+3\left\lceil\frac{n-3k-2-3w}{3}\right\rceil\right) & , & ,if\ n-3k-2\equiv 1(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k-2}{3}\right\rceil-1}\left(\left\lceil\frac{n-3k-2-3w}{3}\right\rceil\right) & , & if\ \ n-3k-2\equiv 2(mod\ 3)\end{matrix}\right\}$$

Then $S^*(dom\ (K\,,n_r\,))=\left\{\begin{matrix}\mu^2 & ,\ if\ n_r=1 \\ 2\mu^2 & ,if\ n_r>1\end{matrix}\right\}$; n_r is an odd number

(III) If $$\mu_1=\left\{\begin{matrix} 1 & , & if\ n-3k\equiv o(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k}{3}\right\rceil-2}\frac{1}{2}\left(\left\lceil\frac{n-3k-3w}{3}\right\rceil^2+3\left\lceil\frac{n-3k-3w}{3}\right\rceil\right) & , & ,if\ n-3k\equiv 1(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k}{3}\right\rceil-1}\left(\left\lceil\frac{n-3k-3w}{3}\right\rceil\right) & , & if\ \ n-3k\equiv 2(mod\ 3)\end{matrix}\right\}$$

and $$\mu_2=\left\{\begin{matrix} 1 & , & if\ n-3k-1\equiv o(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k-1}{3}\right\rceil-2}\frac{1}{2}\left(\left\lceil\frac{n-3k-1-3w}{3}\right\rceil^2+3\left\lceil\frac{n-3k-1-3w}{3}\right\rceil\right) & & ,if\ n-3k-1\equiv 1(mod\ 3) \\ \prod_{w=0}^{\left\lceil\frac{n-3k-1}{3}\right\rceil-1}\left(\left\lceil\frac{n-3k-1-3w}{3}\right\rceil\right) & , & if\ \ n-3k-1\equiv 2(mod\ 3)\end{matrix}\right\}$$

then $S^*(\gamma\ (K\,,n_r\,))=2\mu_1\mu_2$; n_r is an even number

Proof:

(I) We place the first piece of rooks in $s_{n\,,0}$, since in this cell we obtain maximal N_r.This rook bisects the chessboard into two equilateral triangles, the two triangles are of size $n-2$ as shown in Figure 14; $n=8$.

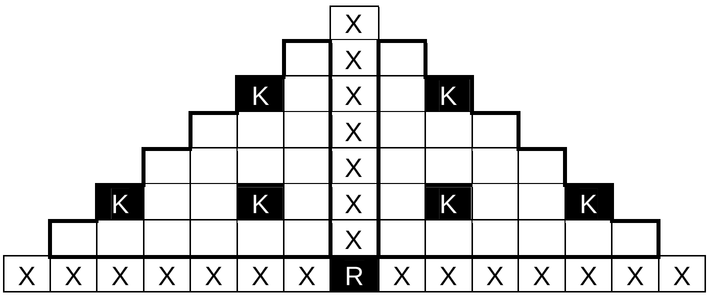


Figure 14

By using Corollary 3.3.3, we obtain $\gamma(K)=\dbinom{\left\lceil\frac{n-2}{3}\right\rceil+1}{\left\lceil\frac{n-2}{3}\right\rceil-1}$ for each one of the two triangles. Thus $\gamma(K\,,1)=2\dbinom{\left\lceil\frac{n-2}{3}\right\rceil+1}{\left\lceil\frac{n-2}{3}\right\rceil-1}$. We must put the second piece of rook in cell such that it has maximal N_r so that we don't make more partition, so one of the two cells $s_{n-1,-1}$ or $s_{n-1,1}$ is suitable for this. Now the cells which are not attacked by these two rooks in the isosceles triangular chessboard are divided into two equilateral triangles of different sizes. One of the two triangles is of size $n-4$ and the other is of size $n-3$ (see Figure 15; $n=8$). By using Corollary 3.3.3, we obtain

$\gamma(K\,,2) = \binom{\left\lceil\frac{n-3}{3}\right\rceil+1}{\left\lceil\frac{n-3}{3}\right\rceil-1} + \binom{\left\lceil\frac{n-4}{3}\right\rceil+1}{\left\lceil\frac{n-4}{3}\right\rceil-1}$.

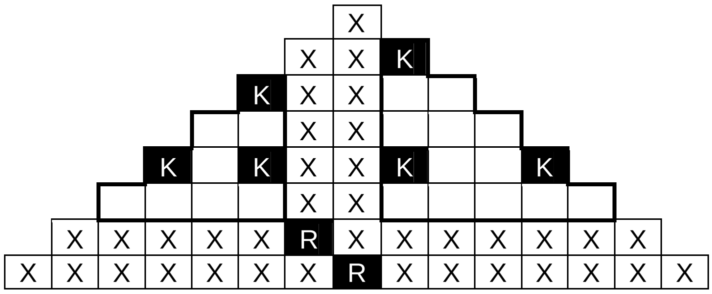


Figure 15

In the case $n_R = 2k+1$; $k = 0,1,\ldots,\left\lceil\frac{n-\left\lfloor\frac{n+1}{3}\right\rfloor-3}{2}\right\rceil$, the cells which are not attacked by these number of rooks constitute two equilateral triangles of size $(n-3k-2)$. From corollary 3.3.3, we obtain $\gamma(K\,,n_r) = 2\binom{\left\lceil\frac{n-3k-2}{3}\right\rceil+1}{\left\lceil\frac{n-3k-2}{3}\right\rceil-1}$.

In case $n_r = 2k$; $k = 1,2,\ldots,\left\lceil\frac{n-\left\lfloor\frac{n+1}{3}\right\rfloor-2}{2}\right\rceil$, the cells which are not attacked by this number of rooks constitute two equilateral triangles of different sizes. One of the two triangles is of size $n-3k$, and the other is of size $n-3k-1$. Again by using Corollary 3.3.3, we obtain $\gamma(K\,,n_r) = \binom{\left\lceil\frac{n-3k}{3}\right\rceil+1}{\left\lceil\frac{n-3k}{3}\right\rceil-1} + \binom{\left\lceil\frac{n-3k-1}{3}\right\rceil+1}{\left\lceil\frac{n-3k-1}{3}\right\rceil-1}$.

(II) From I where n_r is an odd number, we put the first rook in the cell $s_{n,0}$. This rook bisects the chessboard into two equilateral triangles of size $n-2$ (since $k=0$). We get the result in this case by using Corollary 3.3.3. (2), and we obtain the total solution of this case. If we put three rooks such that we put the first in $s_{n,0}$ then there are two possible ways to put the second piece of rook in $s_{n-1,-1}$ or in $s_{n-1,1}$. The suitable cell for the third piece depends on the position of the second piece. Thus we put the third piece in the cell $s_{n-2,-1}$ if the second piece is in $s_{n-1,1}$ (see Figure 16). We continue with the same manner if the number of rooks is more than three pieces. Hence the total solution is $S^*\big(\gamma\,(K\,,n_r\,)\big) = \left\{\begin{matrix}\mu^2 & ,\ if\ n_r = 1\\ 2\mu^2 & ,if\ n_r > 1\end{matrix}\right\}$.

(III) If n_r is an even number and $n > 2$, we put two pieces in the chessboard. From I, these pieces divide the isosceles triangular into two equilateral triangles with different sizes $(n-3k)$ and $(n-3k-1)$. There are two possibilities to put the two pieces in $(s_{n,0}$ and $s_{n-1,1})$ or $(s_{n,0}$ and $s_{n-1,-1})$**.** The other piece of rook doesn't change the total number of methods. Using Corollary 3.3.3. (II). The total solution in this case is given by $S^*\big(\gamma\,(K\,,n_r\,)\big) = 2\mu_1\mu_2$.

The following example is illustrating the above theorem.

Example 4.1.2. If $n = 7$ and $n_r = 3$ then $k = 1$

$\gamma(K\,,3) = 2$, $S^*\big(\gamma\,(\,K\,,n_r\,)\big) = 1$ (see Figure 16)

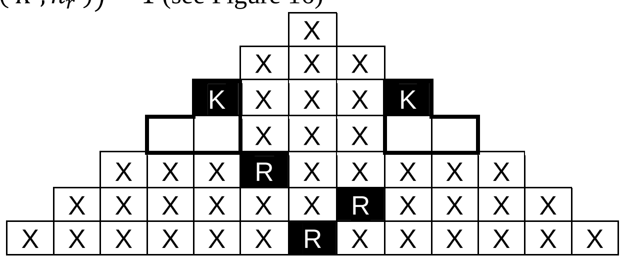


Figure 16

4.2 Domination of kings with a fixed number of bishops

We denote the domination number of kings with a fixed number of bishops n_b by $\gamma(K\,,n_b)$.

Theorem 4.2.1.

(I) $$\gamma(K, n_b) = \begin{cases} \dfrac{\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil}{4}\left(\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil + 2\right), & if \ \left\lceil \frac{2(n-n_b)-1}{3} \right\rceil \ is\ even \\ \dfrac{\left(\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil + 1\right)^2}{4}, & if \ \left\lceil \frac{2(n-n_b)-1}{3} \right\rceil \ is\ odd \end{cases}$$

(II)

$$S^*(\gamma(K, n_b)) = \begin{cases} \prod_{w=0}^{\left\lceil \frac{n-n_b}{3} \right\rceil - 1} \left\lceil \frac{2n-2n_b-(3+6w)}{3} \right\rceil & if\ n-n_b \equiv 0 (mod\ 3) \\ \prod_{w=1}^{\left\lceil \frac{n-n_b}{3} \right\rceil - 1} (2w^2+w) & , if\ n-n_b \equiv 1 (mod\ 3) \\ 1 & if n-n_b \equiv 2 (mod\ 3) \end{cases}$$

Proof:

(I) We place the first piece of bishops in $s_{1,0}$, since in this cell we obtain the maximal N_P. This bishop attacks the two equal sides of the chessboard and the cells which are not be attacked by this piece take the form of isosceles triangular chessboard of size $n-1$. By the same manner for the other pieces of bishops we find that the cells not attacked by these pieces again take the form of isosceles triangular chessboard of size $n-n_b$; $n_b \leq n-2$ and so on. We put the last piece of bishops in cell $s_{n-1,0}$, and one king will be placed in the remaining cell $s_{n,0}$. By using Theorem 3.3.1, we obtain that

$$\gamma(K, n_b) = \begin{cases} \dfrac{\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil}{4}\left(\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil + 2\right), & if \ \left\lceil \frac{2(n-n_b)-1}{3} \right\rceil \ is\ even \\ \dfrac{\left(\left\lceil \frac{2(n-n_b)-1}{3} \right\rceil + 1\right)^2}{4}, & if \ \left\lceil \frac{2(n-n_b)-1}{3} \right\rceil \ is\ odd \end{cases}$$

(II) From I, if we put n_b pieces of bishops as mentioned above, we get a triangle of size $n-n_b$, and the method of distribution of the fixed pieces is unique. Hence, the result.

The following example is illustrating the above theorem.

Example 4.2.2. Let $n = 10, n_b = 3$

$\gamma(3, K) = 9$, $S^*(\gamma(K, n_b)) = 30$, (see Figure 17 ; $n = 10$)

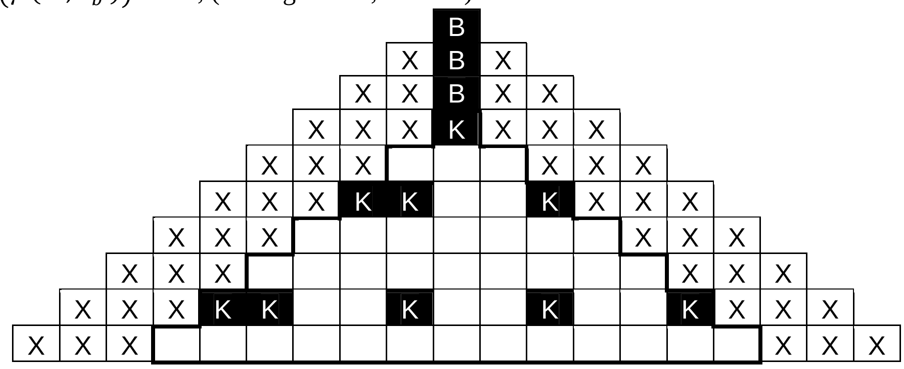


Figure 17

4.3. Domination of rooks with a fixed number of bishops

We denote the domination number of rooks with a fixed number of bishops n_b by $\gamma(R, n_b)$.

Theorem 4.3.1.

(I) $\gamma(R, n_b) = (n-n_b) - \left\lfloor \frac{n-n_b+1}{3} \right\rfloor$

(II) $\left((n-n_b) - \left\lfloor \frac{n-n_b+1}{3} \right\rfloor\right)! \leq S(\gamma(R, n_b)) \leq \left(2\left\lfloor \frac{n-n_b+1}{3} \right\rfloor + 1\right)!$

Proof:

(I) We put the first piece of bishops in our chessboard such that we obtain maximal N_P, the suitable place to do this is the cell $s_{1,0}$.This piece attacks the external (longest) sides of the triangle as shown in Figure 18 by putting "x" in the cells , i.e. we obtain the triangle of size $n-1$. Now we put the second piece of bishops in the cell $s_{2,0}$ and obtain the non-attacked triangle of size $n-2$. This work could be repeated until we reach the cell $s_{n-1,0}$, and then we have to put a rook in the cell $s_{n,0}$. If we have n_b bishops then according to Theorem 3.3.1, we have $\gamma(R, n_b) = (n-n_b) - \left\lfloor \frac{n-n_b+1}{3} \right\rfloor$. Figure 18 shows the case when $n = 9, n_b = 1$.

(II) From I, if we put n_b pieces as mentioned above, we get a triangle of size $n-n_b$. Thus the number of methods of distributing the fixed number of pieces is unique, according to Theorem 3.1.1, we have

$$\left((n-n_b) - \left\lfloor \frac{n-n_b+1}{3} \right\rfloor\right)! \leq S(\gamma(R, n_b)) \leq \left(2\left\lfloor \frac{n-n_b+1}{3} \right\rfloor + 1\right).$$

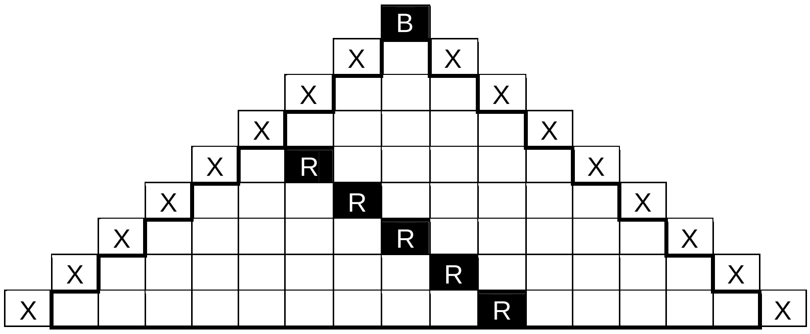


Figure 18

Open problem for two pieces:

Find the general formula of each of the following numbers

$\gamma(\mathrm{R}, \mathrm{n_k})$, $\gamma(\mathrm{B}, \mathrm{n_k})$, $\gamma(\mathrm{B}, \mathrm{n_r})$.

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