Uniform-Pareto Distribution

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Abstract

we introduce (u)niform-Preato distribution U-PD, we discusses some of its properties, distribution, probability density, reliability function, hazard, reserved hazard functions, moments, mode median and its order statistics. Furthermore, the study estimates the shape parameter. We also introduce the simulation study about the estimation of the parameter and the survival function and the application using the data about "spina bifida" disease that the name of the most common birth defect in Babylon province.

Keywords: uniform distribution, Pareto distribution, Uniform-Pareto Distribution, moments.

الخلاصة

في هذا البحث اوجدنا توزيع بأسم توزيع المنتظم –باريتو مع مناقشة بعض من خواصه مثلا دالة الكثافة الاحتمالية، دالة البقاء، دالة الخطورة، العزوم، الوسط، الوسيط، والاحصاءات المرتبة وبعد ذلك درسنا تقدير معلمة الشكل وتقدير دالة البقاء لمرض الصلب المشقوق الذي يصيب الجنين والذي يعتبر كتشوه خلقي لمواليد محافظة بابل .

الكلمات المفتاحية: التوزيع المنتظم، توزيع باريتو، التوزيع المنتظم –باريتو، العزوم.

1.Introduction

For the importance of modelling data of realistic phenomena in real life, there have been made many attempts to define new families that extend well-known distributions, which are intended to generalize well-known distributions.

The aim of this paper is to define a distribution, uniform- Pareto distribution, U-PD, with using the simulation and application.

The Beta-G (Eugene *et al.* (2002) presented the Beta-generated family of distributions with CDF

$$G(x) = \int_0^{F(x)} b(t)\ dt, \qquad (1.1)$$

where $b(t)$ is the pdf of the beta random variable and $F(x)$ is the CDF of any random variable. The corresponding pdf to (1) is given by

$$g(x) = \frac{1}{B(\alpha,\beta)} f(x)F^{\alpha-1}(x)(1 - F(x))^{\beta-1}, \alpha,\beta > 0. \qquad (1.2)$$

Beta-Pareto has been presented by Akinsete, Famoye and Lee (2008), Beta generalized exponential we presented by Barreto-Souza, Santos, and Cordeiro (2009), Beta-half-Cauchy was presented by Cordeiro, and Lemonte (2011), while Beta Generalized Logistic was constructed by Morais, Cordeiro, and Audrey (2011), Beta

–hyperbolic Secant(BHS) was presented by Mattheas, David (2007), Beta Fre´chet was proposed by Nadarajah, and Gupta (2004), Beta normal distribution and its application was constructed by Eugene, *et al*. (2002) and Beta exponential by Nadarajah, and Kotz, (2004).

Uniform Exponential Distribution (UED) and Exponential Pareto distribution(EPD) were introduced by Abed Al-Kadim and Abdalhussain Boshi (2013) have the form

$$F(x) = \int_a^{b-(1-b)F^{\#}(x;\lambda)} f^*(x)\ dx. \tag{1.3}$$

Where $F^{\#}(x;\lambda)$is the c.d.f. of the exponential distribution and $f^*(x)$ is the PDF of the continuous uniform distribution and the form

$$F_{e,p}(x) = \int_a^{\frac{1}{1-F^{\#}(x)}} f^P(x)\ dx \tag{1.4}$$

where $F^{\#}(x) = 1 - \left(\frac{m}{x}\right)^{\theta}$ is the pareto distribution, and $f^P(x)$ is the exponential distribution.

The class Marshall-Olkin (MO-G) of distributions was defined by (Marshall and Olkin 1997), while the Kumaraswamy-G (Kw-G) by, ((Cordeiro and Castro, 2011), the McDonald-G (Mc-G) by (Alexander *et al.* 2012), the gamma-G by (Zografos and Balakrishnan 2009, (the transformer (T-X) by (Alzaatreh *et al.* 2013), the Weibull-G by (Bourguignon *et al*., and the exponentiated half-logistic by (Cordeiro *et al.* 2014.).

In this paper, we introduce (u)niform-Preato distribution U-PD, and discusses some of its properties, distribution, probability density, reliability function, hazard, reserved hazard functions, moments, mode median and its order statistics. Furthermore, the study estimates the shape parameter. Finally, it introduce the simulation study about the estimation of the parameter and the survival function and the application using the data about "spina bifida" disease which is the name of the most common birth defect in Babylon province.

2.The Main Results

Definition (2.1) **Uniform-F Class Distribution**

Let X be a random variable has a distribution from, **Uniform-F Class Distribution**, if its distribution function is defined as follows:

$$\mathrm{G}(x) = \int_a^{bF(x)} \frac{dx}{b-a} \tag{2.1}$$

where $F(x)$ is a c.d.f. of any other distribution, for $-\infty < a \leq b < \infty$.

Definition (2.2) **Uniform-Pareto Distribution (U-PD**)

Let X be a random variable has a distribution from, Uniform- Pareto Distribution, if its distribution function is defined as,

$$G(x) = 1 - \frac{b(\frac{m}{x})^{\theta}}{b-a}\ , \tag{2.2}$$

For $m\left(\frac{b-a}{b}\right)^{-1/\theta} < x < \infty$,

and the scale parameters $-\infty < a \leq b < \infty$, and the shape parameter $\theta > 0$.

Figure 1, shows the shape of the c.d.f. of **U-PD** at different values of the parameters. Is monotone non-decreasing and then it turns to be stable at 1.

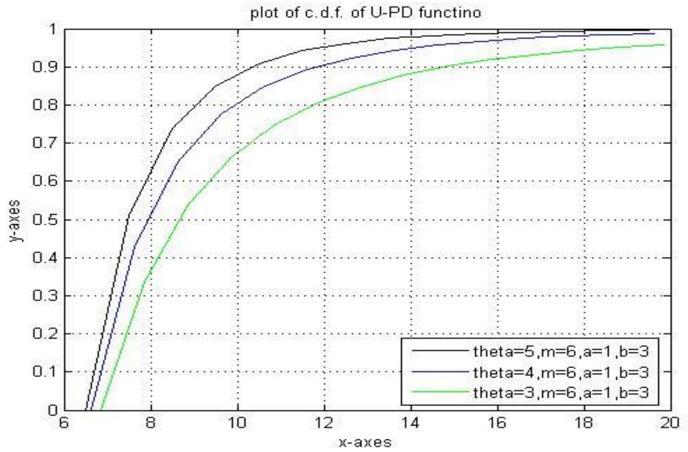

Figure 1 The c.d.f. of U-ParD for $\boldsymbol{\theta} = 1,1.5,2.5$, $\boldsymbol{m} = 2,3$, $\boldsymbol{a} = 1$, $\boldsymbol{b} = 2,3$

Remark (2.1)

If G(x) = 0

1- $\frac{b(\frac{m}{x})^{\theta}}{b-a} = 0$

$b - a - b(\frac{m}{x})^{\theta} = 0$

$\left(\frac{m}{x}\right)^{\theta} = \frac{b-a}{b} \quad \rightarrow \sqrt[\theta]{\frac{b-a}{b}} = \frac{m}{x}$

x = $m\left(\frac{b-a}{b}\right)^{-1/\theta}$,when G(x) =0

If G(x) = 1

1- $\frac{b(\frac{m}{x})^{\theta}}{b-a} = 1$

$\frac{b(\frac{m}{x})^{\theta}}{b-a} = 0$

b$\left(\frac{m}{x}\right)^{\theta} = 0 \rightarrow \left(\frac{m}{x}\right)^{\theta} = 0.$

If x=∞ $\rightarrow \left(\frac{m}{x}\right)^{\theta} = 0$

$m\left(\frac{b-a}{b}\right)^{-1/\theta} < x < \infty$

Now the pdf of U-PD is defined as

$$g(x; m, \theta, a, b) = \frac{b}{b-a}\frac{\theta m^{\theta}}{x^{\theta+1}} \qquad (2.3)$$

For $m\left(\frac{b-a}{b}\right)^{-1/\theta} < x < \infty$, and the scale parameters $-\infty < a \le b < \infty$, and the shape parameter $\theta > 0$. It seems as a weighted Pareto

$$g(x; m, \theta, a, b) = W(a.b)f(x; m.\theta) \qquad (2.4)$$

where $W(a.b) = \frac{b}{b-a} = fixed$

The following figures are about the p.d.f.

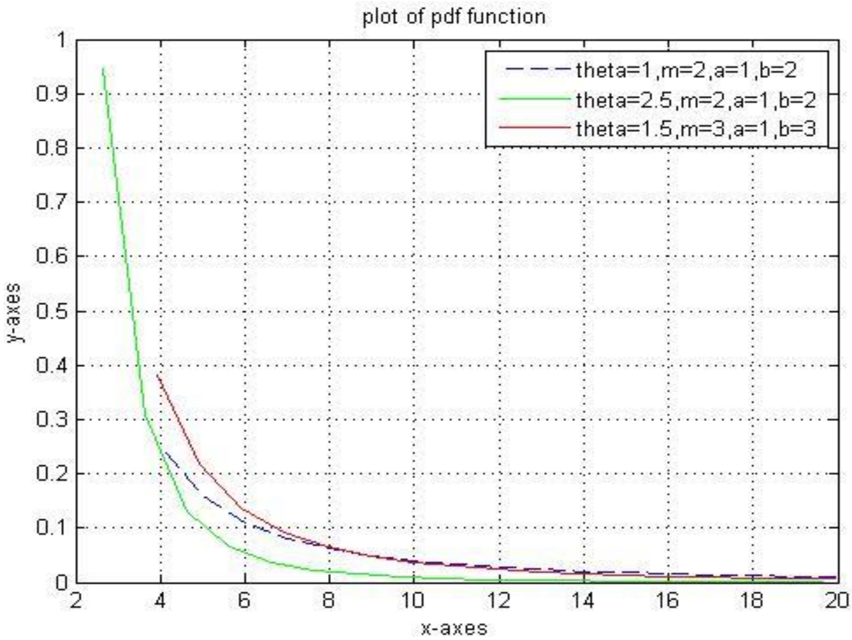

Figure 2. The p.d.f. of U-PD at θ=1,1.5,2.5, $m = 2{,}3, a = 1, b = 2{,}3$

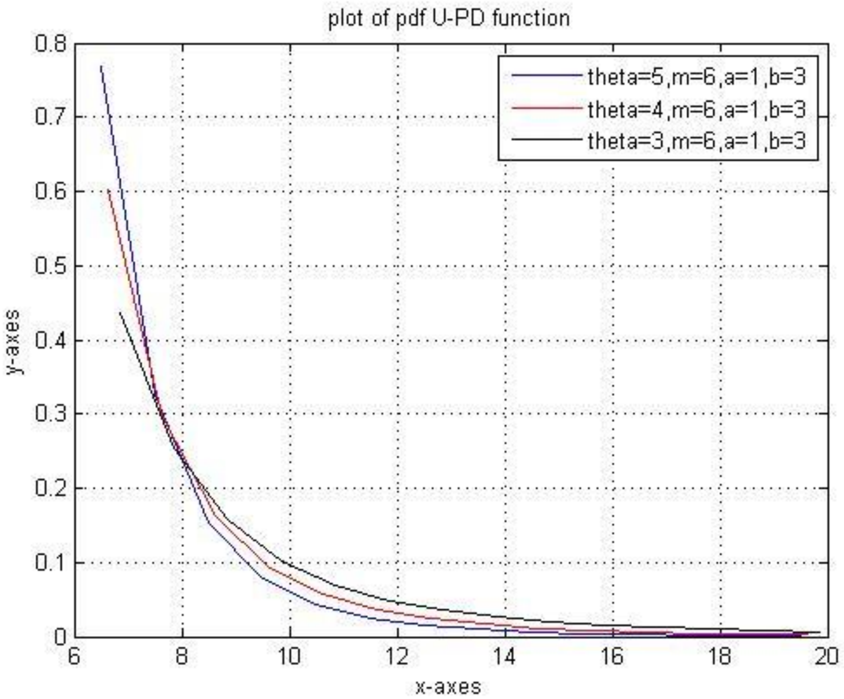

Figure 3 The p.d.f. of U-ParD at $\boldsymbol{\theta = 5, 4, 3}, \boldsymbol{m} = 6, \boldsymbol{a} = 1, \boldsymbol{b} = 3$

These figures, Figure 2, Figure 3 show the shape of the p.d.f. of **U-PD** at different values of the parameters. It is monotonous non-increasing.

Remark(2.2) Let us prove that the function(2.3) is p.d.f. In fact:

$$g(x; m.\theta.a.b) > 0$$

1) Since $a.b > 0$, $\frac{b}{b-a} > 0$

$$m, \theta > 0 \; and \; x > 0 \quad \rightarrow \frac{\theta m^{\theta}}{x^{\theta+1}} > 0.$$

Then $g(x;\, m.\theta.a.b) > 0$

Then let us prove

2) $\int_z^{\infty} g(x; m.\theta.a.b)\, d(x) = 1$, $z = m\,(\frac{b-a}{b})^{-1/\theta}$. In fact

$$\int_z^{\infty} \frac{b}{b-a} \frac{\theta m^{\theta}}{x^{\theta+1}}\, d(x)$$

$=\frac{b}{b-a}\int_z^{\infty} \frac{\theta m^{\theta}}{x^{\theta+1}} d(x)$

$=\frac{b\theta m^{\theta}}{b-a}\int_z^{\infty} \frac{1}{x^{\theta+1}} d(x)$

$=\frac{b\theta m^{\theta}}{b-a}\int_{z}^{\infty} x^{-\theta-1}\, d(x)$

$=\frac{b\theta m^{\theta}}{b-a}\left[\frac{x^{-\theta}}{-\theta}\right]_{z}^{\infty}$

$=\frac{b\theta m^{\theta}}{b-a}\left[\frac{-1}{\theta x^{\theta}}\right]_{z}^{\infty}$

$= 0+\frac{b\theta m^{\theta}}{b-a}\left[\frac{1}{\theta\left(m(\frac{b-a}{b})^{-1/\theta}\right)^{\theta}}\right]$

$=\frac{bm^{\theta}\theta}{b-a}\,\frac{1}{\theta m^{\theta}}\,\frac{b-a}{b}=1$ ■

Proposition (2.1)

The rth central moment about the origin, and the rth central moment about the mean of U-PD are as follows

1- $E(X)^r = \frac{b^{r/\theta}}{(b-a)^{r/\theta}}\,\frac{\theta m^r}{\theta-r}$ (2.5)

2- $E(X-\mu)^r = \frac{bm^{\theta}\theta}{b-a}\sum_{j=0}^{r} C_j^r\,(-\mu)^j\,\frac{m^{r-j-\theta}(\frac{b-a}{b})}{\theta-r-j}$ (2.6)

For $r = 1,2,3, \ldots$

Proof

1- $E(X)^r = \int_{w}^{\infty} x^r\, g(x; m.\theta.a.b)\, dx$

$=\int_{w}^{\infty} x^r\,\frac{b}{b-a}\frac{\theta m^{\theta}}{x^{\theta+1}}\, dx$

$=\frac{bm^{\theta}\theta}{b-a}\int_{w}^{\infty} x^{-\theta+r-1}dx$

$=\frac{bm^{\theta}\theta}{b-a}\left[\frac{x^{-\theta+r}}{-\theta+r}\right]_{w}^{\infty}$ when w=$m(\frac{b-a}{b})^{\frac{-1}{\theta}}$

$=0+\frac{bm^{\theta}\theta}{b-a}\left(\frac{\left(m(\frac{b-a}{b})^{-1/\theta}\right)^{-\theta+r\theta}}{\theta-r}\right)$

$= \frac{b\, m^{\theta}\theta m^{-\theta+r}\, b^{\frac{-\theta+r}{\theta}}}{(b-a)(\theta-r)(b-a)^{\frac{r}{\theta}}}$

$= \frac{b^{r/\theta}}{(b-a)^{r/\theta}}\,\frac{\theta m^r}{(\theta-1)}$

2- $E(X-\mu)^r = \int_{w}^{\infty}(X-\mu)^r\, g(x)dx$

When $w = m(\frac{b-a}{b})^{\frac{-1}{\theta}}$

$=\int_{w}^{\infty}(X-\mu)^r\,\frac{b}{b-a}\frac{\theta m^{\theta}}{x^{\theta+1}}dx$

$$=\frac{b\theta m^{\theta}}{b-a}\int_{w}^{\infty}\frac{(X-\mu)^{r}}{x^{\theta+1}}\,dx$$

using (Binomial theorem)

$$(X-\mu)^{r}=\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j} \tag{2.7}$$

Then

$$\mathrm{E}(X-\mu)^{r}=\frac{b\theta m^{\theta}}{b-a}\int_{w}^{\infty}\frac{\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}}{x^{\theta+1}}\,dx$$

$$=\frac{b\theta m^{\theta}}{b-a}\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}\int_{w}^{\infty}x^{r-j}\;x^{-\theta-1}\,dx$$

$$=\frac{b\theta m^{\theta}}{b-a}\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}\int_{w}^{\infty}x^{r-j-\theta-1}\,dx$$

$$=\frac{b\theta m^{\theta}}{b-a}\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}\left[\frac{x^{r-j-\theta}}{r-j-\theta}\right]_{m(\frac{b-a}{b})^{\frac{-1}{\theta}}}^{\infty}=$$

$$\frac{b\theta m^{\theta}}{b-a}\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}\left[\frac{\left(m(\frac{b-a}{b})^{\frac{-1}{\theta}}\right)^{r-j-\theta}}{r-j-\theta}\right]$$

$$=\frac{b\theta m^{\theta}}{b-a}\sum_{j=0}^{r}C_{j}^{r}(-\mu)^{j}\,(X)^{r-j}\left[\frac{m^{r-j-\theta}(\frac{b-a}{b})^{\frac{\theta+j-r}{\theta}}}{r-j-\theta}\right] \qquad \blacksquare$$

Remark (2.4)

When r=1 from proposition (1),1 we have

$$E(X)=\frac{b^{1/\theta}}{(b-a)^{1/\theta}}\,\frac{\theta m}{\theta-1} \tag{2.8}$$

And r=2 then

$$E(X)^{2}=\frac{b^{2/\theta}}{(b-a)^{2/\theta}}\,\frac{\theta m^{2}}{\theta-2} \tag{2.9}$$

Then

$$Var(X)=E(X)2-(E(X))^{2}$$

$$\sigma^{2}=\frac{b^{2/\theta}}{(b-a)^{2/\theta}}\,\frac{\theta m^{2}}{\theta-2}-\left(\frac{b^{1/\theta}}{(b-a)^{1/\theta}}\,\frac{\theta m}{\theta-1}\right)^{2} \tag{2.10}$$

$$=\frac{b^{2/\theta}\theta m^{2}\left[\left((b-a)^{1/\theta}(\theta-1)\right)^{2}-(b-a)^{2/\theta}(\theta-2)\,\theta\right]}{(b-a)^{2/\theta}(\theta-2)\,\left((b-a)^{1/\theta}(\theta-1)\right)^{2}}$$

Proposition (2.2)

The coefficients of variation, coefficients skewness, and coefficients kurtosis of U_PD are respectively as follows:

$$CV=\frac{\sqrt{\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu}}{\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}(\theta-1)}} \qquad (2.11)$$

$$CS=\frac{\frac{b^{3/\theta}\theta m^3}{(b-a)^{3/\theta}(\theta-3)}-3\mu\frac{b^{2/\theta}\theta m^2}{(b-a)^{2/\theta}(\theta-2)}+2\mu^2\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}(\theta-1)}-\mu^2}{\left(\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu^2\right)^3} \qquad (2.12)$$

$$CK=\frac{\frac{b^{4/\theta}\theta m^4}{(b-a)^{4/\theta}(\theta-4)}-4\mu\frac{b^{3/\theta}\theta m^3}{(b-a)^{3/\theta}(\theta-3)}+6\mu^2\frac{b^{2/\theta}\theta m^2}{(b-a)^{2/\theta}(\theta-2)}-4\mu^3\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}(\theta-1)}+\mu^4}{\left(\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu^2\right)^4} \qquad (2.13)$$

Proof

We have

$$E(X) = \frac{b^{1/\theta}}{(b-a)^{1/\theta}}\frac{\theta m}{\theta-1}$$

And

$$E(X-\mu)^2 = E(X)^2 - 2\mu E(X) - \mu^2 = \sigma^2$$

$$E(X-\mu)^3 = E(X)^3 - 3\mu E(X)^2 + 2\mu^2 E(X)^1 - \mu^3$$

$$E(X-\mu)^4 = E(X)^4 - 4\mu E(X)^3 + 6\mu^2 E(X)^2 - 4\mu^3 E(X)^1 + \mu^4$$

Then

$$CV=\frac{\sigma}{\mu}$$

$$=\frac{\sqrt{\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu}}{\frac{b^{1/\theta}\theta m^1}{(b-a)^{1/\theta}(\theta-1)}}$$

$$CS=\frac{E(X-\mu)^3}{\sigma^3}$$

$$=\frac{E(X)^3-3\mu E(X)^2+2\mu^2 E(X)^1-\mu^3}{\sigma^3}$$

$$=\frac{\frac{b^{3/\theta}\theta m^{3}}{(b-a)^{3/\theta}(\theta-3)}-3\mu\frac{b^{2/\theta}\theta m^{2}}{(b-a)^{2/\theta}(\theta-2)}+2\mu^{2}\frac{b^{1/\theta}\theta m^{1}}{(b-a)^{1/\theta}(\theta-1)}-\mu^{2}}{\left(\frac{b^{1/\theta}\theta m^{1}}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu^{2}\right)^{3}} \qquad \blacksquare$$

$$\text{CK}=\frac{E(X-\mu)^{4}}{\sigma^{4}}$$

$$=\frac{E(X)^{4}-4\mu E(X)^{3}+6\mu^{2}E(X)^{2}-4\mu^{3}E(X)^{1}+\mu^{4}}{\sigma^{4}}$$

$$=\frac{\frac{b^{4/\theta}\theta m^{4}}{(b-a)^{4/\theta}(\theta-4)}-4\mu\frac{b^{3/\theta}\theta m^{3}}{(b-a)^{3/\theta}(\theta-3)}+6\mu^{2}\frac{b^{2/\theta}\theta m^{2}}{(b-a)^{2/\theta}(\theta-2)}-4\mu^{3}\frac{b^{1/\theta}\theta m^{1}}{(b-a)^{1/\theta}(\theta-1)}+\mu^{4}}{\left(\frac{b^{1/\theta}\theta m^{1}}{(b-a)^{1/\theta}}\left(\frac{(\theta-1)-2\theta\mu-2\mu}{(\theta-2)(\theta-1)}\right)+\mu^{2}\right)^{4}} \qquad \blacksquare$$

2.6 **The Survival Function**

The survival function of U-PD

$$S(x)=1-G(x)$$

$$=1-\left(1-\frac{b\left(\frac{m}{x}\right)^{\theta}}{b-a}\right)$$

Thus $S(x)=\frac{b\left(\frac{m}{x}\right)^{\theta}}{b-a}$ (2.14)

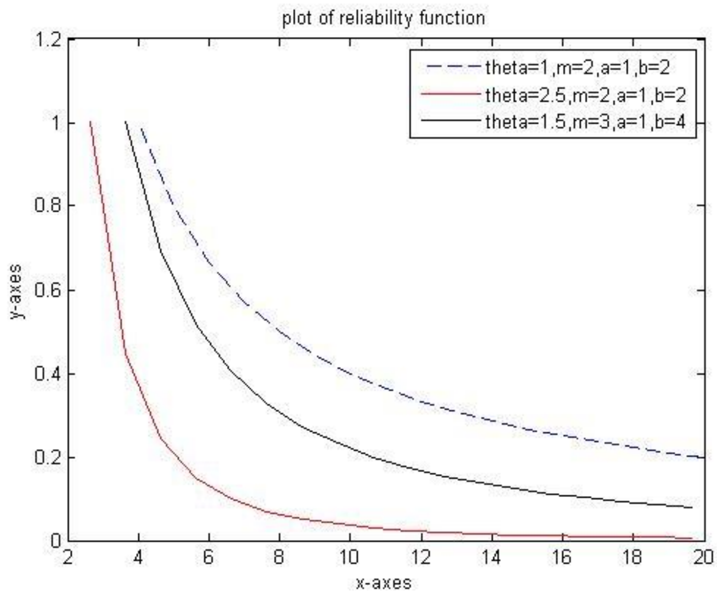

Figure 4 The survival function of U-ParD at $\boldsymbol{\theta=1,1.5,2.5}$

$m=2,3, a=1, b=2,3$

Figure 4 shows the shape of the survival of **U-PD** at different values of the parameters. It monotonous and non-increasing, it seems as the reflex ion of the c.d.f.

2.7 **The Hazard Function**

The hazard function of U-PD

$$h(x) = \frac{g(x)}{S(x)}$$

$$= \frac{x^{\theta}\theta}{x^{\theta+1}}$$

2.8 Conclusions

We can find a distribution named Uniform-Pareto Distribution (U-PD, and discuses some of its properties, distribution, probability density, reliability function, hazard, reserved hazard functions and moments.

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