Exponentiated Transmuted Exponential Distribution

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Abstract
In this research, we introduced the exponentiated transmuted exponential (ETE) distribution. This distribution is more flexible than some distributions so we investigate some of its properties. As well as studying the maximum likelihood estimation of this distribution.

Keywords: Exponentiated transmuted exponential distribution, moment generating function, maximum likelihood estimates.

الخلاصة
في هذا البحث نقدم التوزيع الاسي للاسي المحول. هذا التوزيع هو أكثر مرونة من بعض التوزيعات لذلك ناقشنا بعض الخواص لهذا التوزيع. فضلاً عن دراسة مقدر الامكان الاعظم لهذا التوزيع.

الكلمات المفتاحية التوزيع الاسي للاسي المحول ، الدالة المولدة للعزوم، مقدر الامكان الاعظم .

1. Introduction
The exponential (E) distribution is one of the important families of distributions in lifetime tests .The exponentiated(generalized) family(EF) of distributions were established since the first half of the nineteenth century when Gompertz (1825) and Verhulst (1838, 1845, 1847) used the cumulative distribution function (cdf) (Nadarajah *et al.*, 2015).

$W(t) = \left(1 - \rho e^{-\lambda t}\right)^{\alpha}$, where ρ, α and λ are positive real numbers.

Exponentiated distributions can be obtained by three methods (Moniem et al(2012b)): if we have cdf G(u) of any random variable, then the function $K(u) = (G(u))^{\alpha}, \alpha > 0$ is called an generalized distribution, Using the formula $K(u) = 1 - (1 - G(u))^{\alpha}$ $, \alpha > 0$ and by the transformation $u = \log(x)$,where X is non-negative random variables(Nadarajah, 2005).

Gupta and Kundu (2001) used generalized exponential distribution as an alternative to gamma or Weibull distribution. Sarhan, Baset and Alasbahi (2012 c) introduced the Exponentiated generalized linear exponential distribution and discussed the some statistical properties for the distribution. Flaih et al.(2012a) considered the standard exponentiated inverted weibull distribution (EIW) that exponentiated the standard inverted weibull distribution (IW) and discussed the moments, median and some statistical properties. The Transmuted Exponentiated Gamma Distribution is introduced by Mohamed (2014). Afify et al. (2015a) presented the transmuted Weibull Lomax distribution: Properties and Application by using the transmutation map. Afify et al.(2015b) proposed the Exponentiated transmuted generalized Rayleigh distribution . Exponentiated transmuted Weibull distribution introduced by Ebraheim(2014). Tiensuwan and Pal Presented exponentiated transmuted modified Weibull distribution. The transmuted exponentiated generalized-G family of distributions is proposed by Yousof *et al.*, (2015). Elbatal *et al*., (2013b) introduced the transmuted Quasi Lindley distribution. We introduce the exponentiated transmuted exponential (ETE) distribution ,in this paper. The rest of the search is organized as follows. In Section 2 we introduce exponentiated transmuted exponential distribution (ETED). The Limit for ETED are given in Section 3. In Section 4 we Presented the Statistical the Order Statistics are introduced In Section 5. In Section 6 we presented R´enyi entropy. we demonstrate The parameters estimators ETED in Section 7.We present application in Section 8. Finally, The Conclusions are introduced in section 9.

2. Exponentiated Transmuted Exponential Distribution

Definition 2.1. A random variable X is said to have the transmuted distribution if its cdf is defined as:

$$F^{\#}(x) = (1+\beta)G(x) - \beta(G(x))^2\,,\ |\beta| \le 1 \qquad (1)$$

where $G(x)$ is the cdf of the base distribution. When β = 0, we have the distribution of the base random variable.

Definition 2.2. A random variable X is said to have the exponentiated distribution if its cdf is defined as:

$$F(x;\alpha,\beta,\lambda) = \left(F^{\#}(x)\right)^{\alpha} \qquad (2)$$

The cdf of exponentiated transmuted exponential distribution with parameters α, β and λ is defined by:

$$F(x;\alpha,\beta,\lambda) = \left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{\alpha}\quad ,\quad x > 0, \alpha, \beta, \lambda > 0 \qquad (3)$$

and its pdf is

$$f(x;\alpha,\lambda,\beta) = \alpha\lambda e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{\alpha-1} \qquad (4)$$

where α is a shape parameter, and β, λ are a scale parameters.

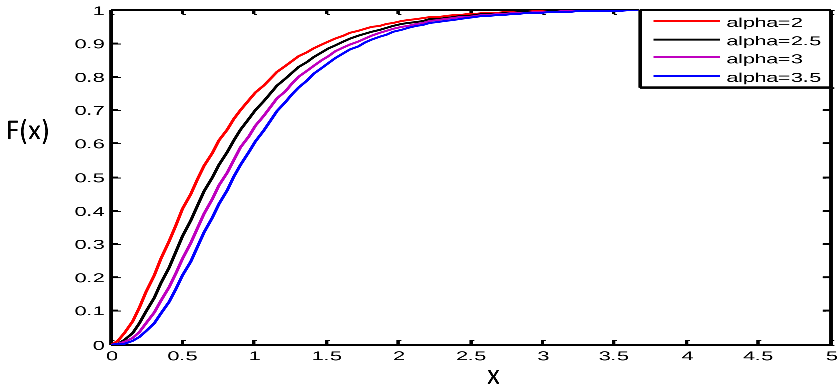

Figure 1: The cdf of ETED with fixed $\beta = \lambda = 1$ and α take the values $(2, 2.5, 3, 3.5)$.

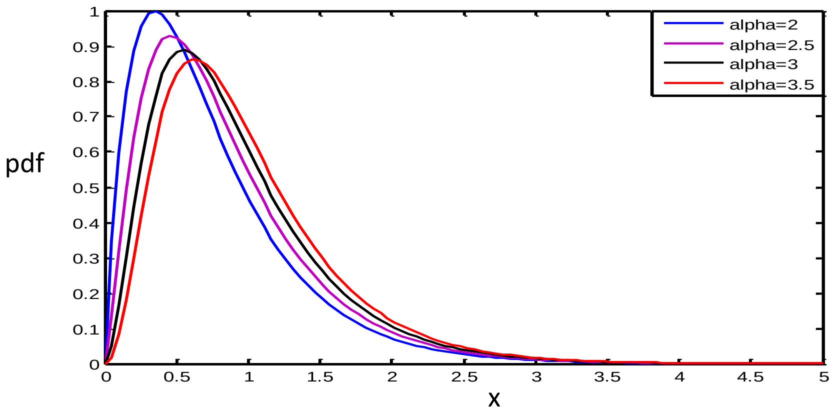

Figure 2: The pdf of ETED with fixed $\beta = \lambda = 1$ and α take the values (2,2.5,3,3.5).

3. Limit of the density function and cumulative function:

The limit of pdf

$$\lim_{x\to 0} f(x,\alpha,\lambda,\beta) = \lim_{x\to 0} \alpha\lambda e^{-\lambda x} \lim_{x\to 0} \left((1-\beta) + 2\beta e^{-\lambda x}\right)$$
$$\times \lim_{x\to 0} \left(1-e^{-\lambda x}\right)^{\alpha-1} \lim_{x\to 0} \left(1+\beta e^{-\lambda x}\right)^{\alpha-1} = 0 \quad (5)$$

and

$$\lim_{x\to \infty} f(x,\alpha,\lambda,\beta) = \lim_{x\to \infty} \alpha\lambda e^{-\lambda x} \lim_{x\to 0} \left((1-\beta) + 2\beta e^{-\lambda x}\right)$$
$$\times \lim_{x\to \infty} \left(1-e^{-\lambda x}\right)^{\alpha-1} \lim_{x\to \infty} \left(1+\beta e^{-\lambda x}\right)^{\alpha-1} = 0 \quad (6)$$

The limit of cdf of ETED:

$$\lim_{x\to 0} F(x,\alpha,\lambda,\beta) = \lim_{x\to 0} \left(1-e^{-\lambda x}\right)^{\alpha-1} \lim_{x\to 0} \left(1+\beta e^{-\lambda x}\right)^{\alpha-1} = 0 \quad (7)$$

and $\lim_{x\to \infty} F(x,\alpha,\lambda,\beta) = \lim_{x\to \infty} \left(1-e^{-\lambda x}\right)^{\alpha-1} \lim_{x\to \infty} \left(1+\beta e^{-\lambda x}\right)^{\alpha-1} = 1$ (8)

4. Statistical Properties

We study statistical properties of the ETE distribution in this section.

4.1 Reliability function

The reliability function $R(x,\alpha,\lambda,\beta)$ of ETE distribution is given by

$$R(x,\alpha,\lambda,\beta) = 1 - F(x,\alpha,\lambda,\beta)$$
$$= 1 - \left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{\alpha} \quad (9)$$

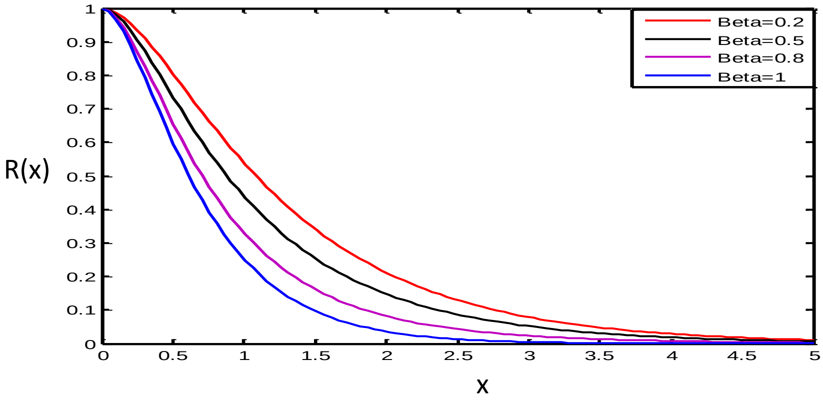

Figure 3: The reliability function of ETED with fixed $\alpha = 2, \lambda = 1$ and β take the values (0.2,0.5,0.8,1).

4.2 Hazard function

The hazard function of ETE distribution is given by:

$$h(x,\alpha,\lambda,\beta) = \frac{f(x,\alpha,\lambda,\beta)}{R(x,\alpha,\lambda,\beta)}$$

$$= \frac{\alpha\lambda e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{\alpha-1}}{1-\left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{\alpha}} \tag{10}$$

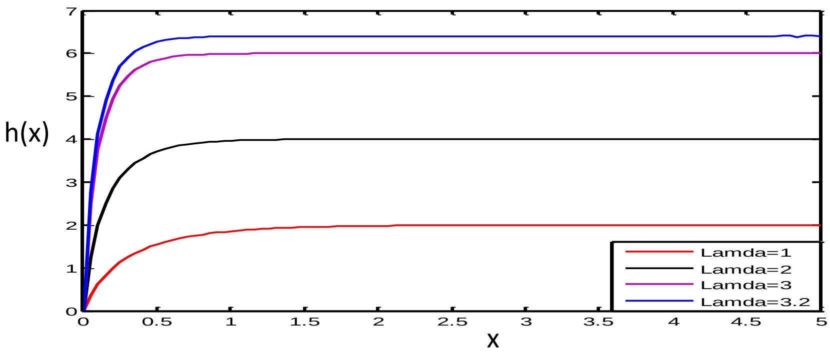

Figure 4: The hazard function of ETED with fixed $\alpha = 2, \beta = 1$ and λ take the values (1,,2,3,3.2).

4.3 Reverse hazard function

The reverse hazard function $\varphi(x,\alpha,\lambda,\beta)$ of ETED is defined as

$$\varphi(x,\alpha,\lambda,\beta) = \frac{f(x,\alpha,\lambda,\beta)}{F(x,\alpha,\lambda,\beta)}$$

$$= \alpha\lambda e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\left(\left(1-e^{-\lambda x}\right)\left(1+\beta e^{-\lambda x}\right)\right)^{-1} \tag{11}$$

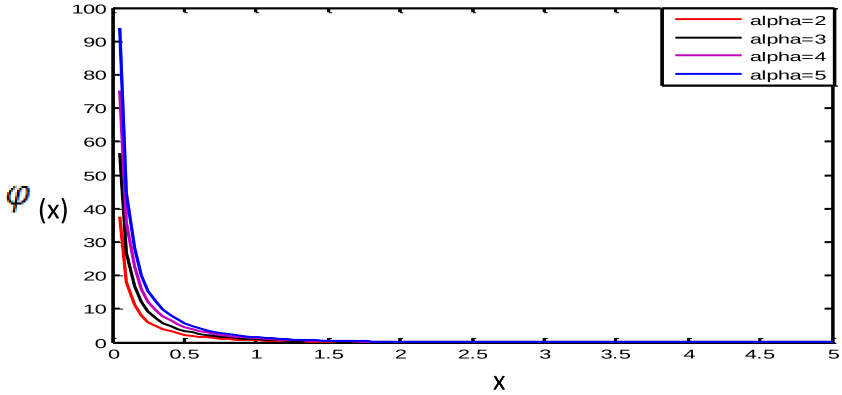

Figure 5: The reverse hazard function function of ETED with fixed $\beta = \lambda = 1$ and α take the values (2,3,4,5).

4.4 Cumulative hazard function

The cumulative hazard function of the ETED is denoted by $H\,(x\; x; \alpha, \lambda, \beta)$ and is defined by

$$H(x; \alpha, \lambda, \beta) = -\ln(1 - F(x, \alpha, \lambda, \beta)) = -\ln\left(1 - \left(\left(1 - e^{-\lambda x}\right)\left(1 + \beta e^{-\lambda x}\right)\right)^{\alpha}\right) \qquad (12)$$

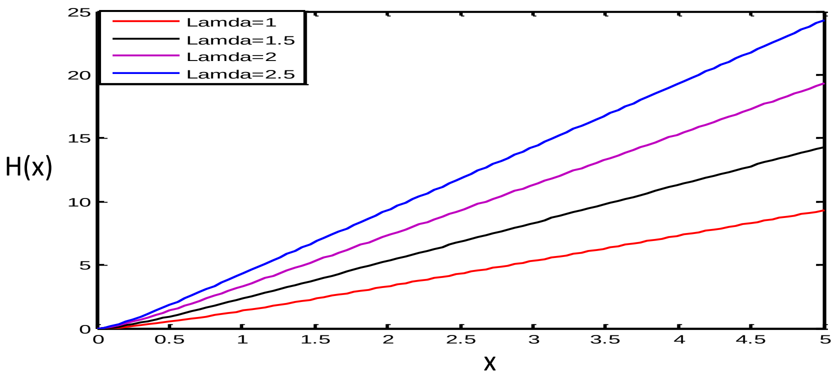

Figure 6: The cumulative hazard function of ETED with fixed α=2,β=1 and λ take the values (1,1.5,2,2.5).

4.5 Moments of The ETE Distribution

We present the moment about the mean and the moment about the origin of the ETED, in this subsection.

Theorem 4.5.1. The r^{th} moment about the mean of ETED is given by:

$$E(X-\mu)^r = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^{k}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{r}{k}\binom{1}{m}$$
$$\times \frac{2^{m}\Gamma(r-k+1)}{(i+j+k+m+1)^{r-k+1}\lambda^{r-k}} \qquad (13)$$

and the moment about the origin is

$$E(X^r) = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i \beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}\frac{2^m\Gamma(r+1)}{(i+j+m+1)^{r+1}\lambda^r}$$

r=1,2,3,…. (14)

Proof:

Using equation (4),the r^{th} moment is given by

$$E(X-\mu)^r = \int_0^{\infty}(x-\mu)^r f(x,\alpha,\lambda,\beta)dx$$
$$= \alpha\lambda\int_0^{\infty}(x-\mu)^r e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\left(1-e^{-\lambda x}\right)^{\alpha-1}$$
$$\times\left(1+\beta e^{-\lambda x}\right)^{\alpha-1}dx \tag{15}$$

using the series expansion of $\left(1-e^{-\lambda x}\right)^{\alpha-1}$, $\left(1+\beta e^{-\lambda x}\right)^{\alpha-1}$ and $(x-\mu)^r$

$$\left(1-e^{-\lambda x}\right)^{\alpha-1} = \sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}e^{-i\lambda x} \tag{16}$$

$$\left(1+\beta e^{-\lambda x}\right)^{\alpha-1} = \sum_{j=0}^{\infty}\binom{\alpha-1}{i}\beta^j e^{-j\lambda x} \tag{17}$$

and $$(X-\mu)^r = \sum_{k=0}^{r}(-1)^k\binom{r}{k}x^{r-k}\mu^k \tag{18}$$

$$\left((1-\beta)+2\beta e^{-\lambda x}\right) = \sum_{m=0}^{1}\binom{1}{m}(1-\beta)^{1-m}(2\beta)^m e^{-m\lambda x} \tag{19}$$

Applying (16) , (17) ,(18)and (19) in (15) ,we get

$$E(X-\mu)^r = \alpha\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1}(-1)^{i+k}2^m\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{r}{k}\binom{1}{m}$$
$$\times\int_0^{\infty}x^{r-k}e^{-(i+j+m+1)\lambda x}dx$$

Hence

$$E(X-\mu)^r = \alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{r}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{r}{k}\binom{1}{m}$$
$$\times\frac{2^m\Gamma(r-k+1)}{(i+j+k+m+1)^{r-k+1}\lambda^{r-k}}$$

Then If $\mu = E(X) = 0$. Hence the moment about the origin is

$$E(X^r) = \alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i\beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}\frac{2^m\Gamma(r+1)}{(i+j+m+1)^{r+1}\lambda^r}$$

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If r =1

$$E(X-\mu) = \alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{1}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{k}\binom{1}{m}$$
$$\times\frac{2^m\Gamma(2-k)}{(i+j+k+m+1)^{2-k}\lambda^{1-k}} \tag{20}$$

If r =2

$$E(X-\mu)^2 = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{2}{k}\binom{1}{m} \times \frac{2^m\Gamma(3-k)}{(i+j+k+m+1)^{3-k}\lambda^{2-k}} \quad (21)$$

If $r = 3$

$$E(X-\mu)^3 = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{3}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{3}{k}\binom{1}{m} \times \frac{2^m\Gamma(4-k)}{(i+j+k+m+1)^{4-k}\lambda^{3-k}} \quad (22)$$

If r =4

$$E(X-\mu)^4 = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{4}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{4}{k}\binom{1}{m} \times \frac{2^m\Gamma(5-k)}{(i+j+k+m+1)^{5-k}\lambda^{4-k}} \quad (23)$$

The **Variance** is given by

$$var(X) = E(X-\mu)^2 = \alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{2}{k}\binom{1}{m} \times \frac{2^m\Gamma(3-k)}{(i+j+k+m+1)^{3-k}\lambda^{2-k}} \quad (24)$$

The **Coefficient of Variation** is given by

$$CV = \frac{\sqrt{var(X)}}{\mu(X)} = \frac{\sqrt{\alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{2}{k}\binom{1}{m}\frac{2^m\Gamma(3-k)}{(i+j+k+m+1)^{3-k}\lambda^{2-k}}}}{\alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^{i}\beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}\frac{2^m}{(i+j+m+1)^2\lambda}} \quad (25)$$

The **Coefficient of skewness** is given by

$$CS = \frac{E(X-\mu)^3}{(E(X-\mu)^2)^{\frac{3}{2}}} = \frac{\alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{3}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{3}{k}\binom{1}{m}\frac{2^m\Gamma(4-k)}{(i+j+k+m+1)^{4-k}\lambda^{3-k}}}{\left(\alpha \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{2}{k}\binom{1}{m}\frac{2^m\Gamma(3-k)}{(i+j+k+m+1)^{3-k}\lambda^{2-k}}\right)^{\frac{3}{2}}} \quad (26)$$

The **Coefficient of kurtosis** is given by

$$CK=\frac{E(X-\mu)^4}{(E(X-\mu)^2)^2}$$

$$=\frac{\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{4}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{4}{k}\binom{1}{m}\frac{2^m\Gamma(5-k)}{(i+j+k+m+1)^{5-k}\lambda^{4-k}}}{\left(\alpha\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{2}\sum_{m=0}^{1}(-1)^{i+k}\beta^{j+m}\mu^k(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{2}{k}\binom{1}{m}\frac{2^m\Gamma(3-k)}{(i+j+k+m+1)^{3-k}\lambda^{2-k}}\right)^2} \quad (27)$$

4.6 Moment Generating Function of ETED

In this subsection, we present the moment generating function of ETED.

Theorem 4.6.2 . the moment generating function of ETE distribution is given by:

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i 2^m\beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}\frac{\alpha\lambda}{((i+j+m+1)\lambda-t)} \quad (28)$$

Proof: We start with the well-known definition of the moment generating function given by:

$$M_X(t)=E(e^{tx})=\int_0^{\infty}e^{tx}f(x,\alpha,\lambda,\beta)dx$$

$$=\alpha\lambda\int_0^{\infty}e^{-(\lambda-t)x}\left((1-\beta)+2\beta e^{-\lambda x}\right)\left((1-e^{-\lambda x})(1+\beta e^{-\lambda x})\right)^{\alpha-1}dx$$

$$=\alpha\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i 2^m\beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}$$

$$\times\int_0^{\infty}e^{-((i+j+m+1)\lambda-t)x}dx$$

Hence

$$M_X(t)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{1}(-1)^i 2^m\beta^{j+m}(1-\beta)^{1-m}\binom{\alpha-1}{i}\binom{\alpha-1}{j}\binom{1}{m}\frac{\alpha\lambda}{((i+j+m+1)\lambda-t)}$$

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5. Order Statistics

Let $X_1, X_2, \dots, X_n$ be a random sample of size n from EEPD distribution with cdf $F(x,\boldsymbol{\vartheta})$ and pdf $f(x,\boldsymbol{\vartheta})$ given by (3) and (4) respectively ,where $\boldsymbol{\vartheta}=(\alpha,\lambda,p,\theta)$ .Let $X_{1:n}\le X_{2:n}\le\cdots\le X_{r:n}$ denote the order statistics obtained from this sample.The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x;\boldsymbol{\vartheta}) = \frac{1}{B(r,n-r+1)}\left(F(x;\boldsymbol{\vartheta})\right)^{r-1}\left(1-F(x;\boldsymbol{\vartheta})\right)^{n-r} f(x;\boldsymbol{\vartheta}) \tag{29}$$

Since $0 < F(x;\boldsymbol{\vartheta}) < 1$, for $x > 0$, we can use the binomial expansion of $\left(1-F(x;\boldsymbol{\vartheta})\right)^{n-r}$ given as follows:

$$\left(1-F(x;\boldsymbol{\vartheta})\right)^{n-r} = \sum_{i=0}^{n-r}(-1)^i \binom{n-r}{i}\left(F(x,\theta)\right)^i \tag{30}$$

applying (30) in(29) ,we get

$$f_{r:n}(x;\boldsymbol{\vartheta}) = \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i \binom{n-r}{i} f(x,\theta)\left(F(x,\theta)\right)^{r+i-1}$$

$$= \frac{1}{B(r,n-r+1)}\sum_{i=0}^{n-r}(-1)^i \binom{n-r}{i}\lambda\alpha e^{-\lambda x}\left((1-\beta)+2\beta e^{-\lambda x}\right)$$
$$\times\left((1-e^{-\lambda x})(1+\beta e^{-\lambda x})\right)^{\alpha(r+i)-1} \tag{31}$$

6. R´enyi entropy

Entropy refers to the amount of uncertainty associated with a random variable. It is an important concept in many fields of science, especially theory of communication, physics and probability. The R´enyi entropy is defined by

$$I_R(\delta) = \frac{1}{1-\delta}\log\int_o^\infty (f(x;\alpha,\beta,\lambda))^\delta dx \tag{32}$$

Proposition 6.1. If X is random variable has a ETED, then, the R´enyi entropy is defined by:

$$I_R(\delta) = \frac{1}{1-\delta}$$
$$\times\log\left(\alpha^\delta\lambda^{\delta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\delta}{i}\binom{\delta(\alpha-1)}{j}\binom{\delta(\alpha-1)}{k}\frac{(1-\beta)^{\delta-i}2^i\beta^{i+k}}{(\delta+i+j+k)}\right) \tag{33}$$

Proof:

$$I_R(\delta) = \frac{1}{1-\delta}\log\int_o^\infty (f(x;\alpha,\beta,\lambda))^\delta dx \tag{34}$$

let $A = \int_o^\infty \alpha^\delta\lambda^\delta\, e^{-\lambda\delta x}\left((1-\beta)+2\beta e^{-\lambda x}\right)^\delta$
$$\times\left(1-e^{-\lambda x}\right)^{\delta(\alpha-1)}\left(1+\beta e^{-\lambda x}\right)^{\delta(\alpha-1)}dx$$

$$= \alpha^\delta\lambda^\delta(1-\beta)^{\delta-i}2^i\beta^{i+k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\delta}{i}\binom{\delta(\alpha-1)}{j}\binom{\delta(\alpha-1)}{k}$$
$$\times\int_o^\infty e^{-(\delta+i+j+k)\lambda x}dx$$

$$= \alpha^\delta\lambda^{\delta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^j\binom{\delta}{i}\binom{\delta(\alpha-1)}{j}\binom{\delta(\alpha-1)}{k}\frac{(1-\beta)^{\delta-i}2^i\beta^{i+k}}{(\delta+i+j+k)}$$

Hence

$$I_R(\delta)=\frac{1}{1-\delta}$$
$$\times\, log\left(\alpha^{\delta}\lambda^{\delta-1}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{j}\binom{\delta}{i}\binom{\delta(\alpha-1)}{j}\binom{\delta(\alpha-1)}{k}\frac{(1-\beta)^{\delta-i}2^{i}\beta^{i+k}}{(\delta+i+j+k)}\right)$$

7. Maximum Likelihood Estimators

We determine the maximum likelihood estimates (MLE) of the parameters, in this section. Let $x_1,x_2,\ldots,x_n$ be an i.i.d random of size n from a distribution that has the a pdf $f(x;\boldsymbol{\vartheta})$, $\boldsymbol{\vartheta}=(\alpha,\beta,\lambda)$,The joint density of the sample is, by independence, equal to the product of the marginal densities .

Then the Maximum Likelihood function (L) is The joint probability density function of the random sample $x_1,x_2,\ldots,x_n$ and can be written as,

$$l(\boldsymbol{\vartheta})=f(x_1,x_2,\ldots,x_n;\boldsymbol{\vartheta})=\prod_{i=1}^{n}f(x_i;\boldsymbol{\vartheta}) \qquad (35)$$

The log-likelihood function for the vector of parameters can be written as,

$$\mathrm{L}=\log l(\boldsymbol{\vartheta})=n\log\alpha+n\log\lambda-\lambda\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left((1-\beta)+2\beta e^{-\lambda x_i}\right)$$
$$+(\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)+(\alpha-1)\sum_{i=1}^{n}\log\left(1+\beta e^{-\lambda x_i}\right) \qquad (36)$$

Then by take the partial derivative of L with respect to α,β and λ, respectively, we get:

$$\frac{\partial \mathrm{L}}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)+\sum_{i=1}^{n}\log\left(1+\beta e^{-\lambda x_i}\right) \qquad (37)$$

$$\frac{\partial \mathrm{L}}{\partial\beta}=\sum_{i=0}^{n}\frac{-1+2e^{-\lambda x_i}}{\left((1-\beta)+2\beta e^{-\lambda x_i}\right)}-(\alpha-1)\sum_{i=0}^{n}\frac{e^{-\lambda x_i}}{\left(1+\beta e^{-\lambda x_i}\right)} \qquad (38)$$

$$\frac{\partial \mathrm{L}}{\partial\lambda}=\frac{n}{\lambda}-\sum_{i=1}^{n}x_i-\sum_{i=0}^{n}\frac{2\beta\, x_i e^{-\lambda x_i}}{\left((1-\beta)+2\beta e^{-\lambda x_i}\right)}-(\alpha-1)\sum_{i=0}^{n}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}$$
$$-(\alpha-1)\sum_{i=0}^{n}\frac{\beta x_i e^{-\lambda x_i}}{\left(1+\beta e^{-\lambda x_i}\right)} \qquad (39)$$

The solving of the equations $\frac{\partial \mathrm{L}}{\partial\alpha}=0$, $\frac{\partial L}{\partial\beta}=0$,and $\frac{\partial \mathrm{L}}{\partial\lambda}=0$, yields the maximum likelihood (ML) estimates of α,β and λ as following

$$\hat{\alpha}=\frac{-n}{\sum_{i=1}^{n}\log\left(1-e^{-\lambda x_i}\right)+\sum_{i=1}^{n}\log\left(1+\beta e^{-\lambda x_i}\right)} \qquad (40)$$

And the maximum likelihood estimate of $\beta$and λ , can be obtained by numerical methods. Also when all parameters are unknown, the MLE of them can be obtained by the numerical methods.

8. Application

In this section, we present two applications to real data to illustrate the importance of the ETE distribution The real data set represent the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee, 1992). This data set has recently been studied by Muhammad et al. (2015). The data are: 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

In the following, we compare ETE distribution with other three lifetime models lognormal distribution(LND), log-logistic distribution(LLD) and Exponential distribution(ED).

Table 2 provides the MLEs of the model parameters. The model selection is carried out using the AIC (Akaike information criterion), the BIC (Bayesian information criterion) and the CAIC (consistent Akaike information criteria):

$$\text{AIC}= -2L(\hat{\theta}) + 2p\ ,\ BIC = -2L(\hat{\theta}) + plog(n)\ ,\ \text{CAIC} = -2L(\hat{\theta}) + \frac{2pn}{n-p-1} \qquad (41)$$

where $L(\hat{\theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size.

Table 1. MLEs of their distributions for the cancer data

Distribution	Estimates				
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	μ	σ
LN (α, β)	--	--	--	3.46	1.033
	--	--	--	(0.094)	(0.066)
LL (μ, σ)	1.495	30.984	--	--	--
	(0.175)	(4.290)	--	--	--
E (λ)	--	--	0.0216	--	--
	--	--	--	--	--
ETE (α, β, λ)	1.6459	0.6636	0.0192	--	--
	(0.25)	(1.04)	(0.005)	--	--

Table 2. The AIC, CAIC , BIC values for the Cancer data

Distribution	statistics			
	$\hat{L}$	AIC	BIC	CAIC
LN	- 595.034	1194.067	1194.168	1199.658
LL	- 587.6	1179.199	1184.791	1179.301
E	- 585.1277	1172.26	1175.05	1172.29
ETE	- 581.8153	**1169.63**	1178.02	**1169.84**

Table 2. The AIC, CAIC , BIC values for Breast Cancer data

We noted from Table 2 that ETED model shows lowest values of AIC ,BIC and CAIC among the fitted models: ETED, LND,LLD and ED , suggesting that the ETED shows the best fit, and therefore could be chosen as the best model.

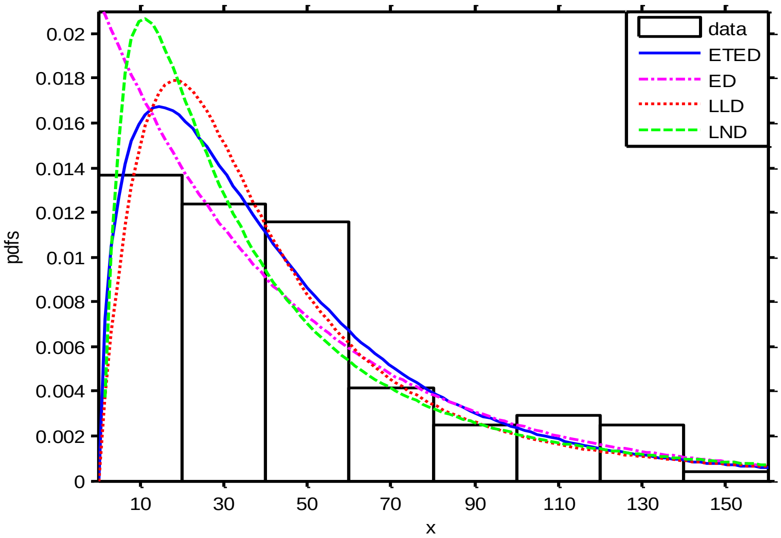

Figure 7. plot of the estimated pdfs

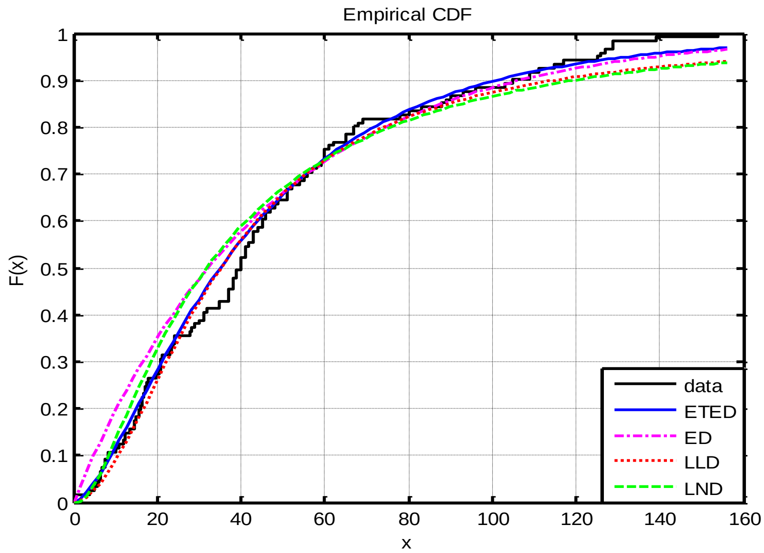

Figure 8. plot of the estimated cdfs

9. Conclusions

In this research we presented the Exponentiated transmuted Exponential distribution. We investigate some of its properties. It is observed that the MLE of the unknown parameters can be obtained numerically.

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