Volume 1 No.6, December 2011 ISSN 2224-3577
International Journal of Science and Technology

http://www.ejournalofsciences.org

Weak Separation Axioms via Ω −Open Set and Ω −Closure Operator

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ABSTRACT

In this paper we introduce a new type of weak separation axioms with some related theorems and show that they are equivalent with these in [1].

Keywords: *Weak separation axioms, weak* $\omega-$ open sets, *weak regular spaces.*

1. INTRODUCTION AND AUXILIARY RESULTS

In this article let us prepare the background of the subject. Throughout this paper, (X,T) stands for topological space. Let A be a subset of X. A point x in X is called *condensation* point of A if for each U in T with x in U, the set $U\cap A$ is uncountable [2]. In 1982 the ω −closed set was first introduced by H. Z. Hdeib in [2], and he defined it as: A is ω *−closed* if it contains all its condensation points and the ω *−open* set is the complement of the ω −closed set. It is not hard to prove: any open set is ω −open. Also we would like to say that the collection of all ω −open subsets of X forms topology on X. The closure of A will be denoted by $cl(A)$, while the intersection of all ω −closed sets in X which containing A is called the ω *−closure* of A, and will denote by $cl_\omega(A)$. Note that $cl_\omega(A)\subset cl(A)$.

In 2005 M. Caldas, T. Fukutake, S. Jafari and T. Noiri [3] introduced some weak separation axioms by utilizing the notions of $\delta-pre$ −open sets and $\delta-pre$ −closure. In this paper we use M. Caldas, T. Fukutake, S. Jafari and T. Noiri [3] definitions to introduce new spaces by using the ω −open sets defined by H. Z. Hdeib in [3], we ecall it $\omega-R_i$ −Spaces $i=0,1,2$, and we show that $\omega-R_0$, $\omega^\star-T_1$ space and ω −symmetric space are equivalent.

For our main results we need the following definitions and results:

Definition 1.1: [4] A space (X,T) is called a *door space* if every subset of X is either open or closed.

Definition 1.2: [1] The topological space X is called $\omega^\star-T_1$ *space* if and only if, for each $x\neq y\in X$, there exist ω −open sets U and V, such that $x\in U, y\notin U$, and $y\in V, x\notin V$.

Lemma 1.3: [1] The topological X is $\omega^\star-T_1$ if and only if for each $x\in X,\{x\}$ is ω −closed set in X.

Definition 1.4: [1] The topological space X is called $\omega^\star-T_2$ *space* if and only if, for each $x\neq y\in X$, there exist two disjoint ω −open sets U and V with $x\in U$ and $y\in V$.

For our main result we need the following property of ω −closure of a set:

Proposition 1.5: Let $\{A_\lambda,\lambda\in\Lambda\}$ be a family of subsets of the topological space (X,T), then

1. $cl_\omega(\cap_{\lambda\in\Lambda}A_\lambda)\subseteq\cap_{\lambda\in\Lambda}cl_\omega(A_\lambda)$.
2. $\cup_{\lambda\in\Lambda}cl_\omega(A_\lambda)\subseteq cl_\omega(\cup_{\lambda\in\Lambda}A_\lambda)$.

Proof:

1. It is clear that $\cap_{\lambda\in\Lambda}A_\lambda\subseteq A_\lambda$ for each $\lambda\in\Lambda$. Then by (4) of Theorem 1.5.3 in [1], we have $cl_\omega(\cap_{\lambda\in\Lambda}A_\lambda)\subseteq cl_\omega(A_\lambda)$ for each $\lambda\in\Lambda$. Therefore $cl_\omega(\cap_{\lambda\in\Lambda}A_\lambda)\subset\cap_{\lambda\in\Lambda}cl_\omega(A_\lambda)$.

Note that the opposite direction is not true . For example consider the usual topology T for $\mathbb{R}$, If $A_i=\left(0,\frac{1}{i}\right), i=1,2,\ldots$, and $\cap_{i\in\mathbb{N}}cl_\omega(A_i)=\{0\}$. But $cl_\omega(\cap_{i\in\mathbb{N}}A_i)=cl_\omega(\emptyset)=\emptyset$. Therefore $\cap_{\lambda\in\Lambda}cl_\omega(A_\lambda)\nsubseteq cl_\omega(\cap_{\lambda\in\Lambda}A_\lambda)$.

2. Since $A_\lambda\subseteq\cup_{\lambda\in\Lambda}A_\lambda$, for each $\lambda\in\Lambda$. Then by (4) of Theorem 1.5.3 in [2], we get $cl_\omega(A_\lambda)\subseteq cl_\omega(\cup_{\lambda\in\Lambda}A_\lambda)$, for each $\lambda\in\Lambda$. Hence $\cup_{\lambda\in\Lambda}cl_\omega(A_\lambda)\subseteq cl_\omega(\cup_{\lambda\in\Lambda}A_\lambda$.

Note that the opposite direction is not true . For example consider the usual topology T for $\mathbb{R}$, If $A_i=\left\{\frac{1}{i}\right\}, i=1,2,\ldots$, $cl_\omega(A_i)=\left\{\frac{1}{i}\right\}$, and $\cup_{i\in\mathbb{N}}cl_\omega(A_i)=\left\{1,\frac{1}{2},\frac{1}{3},\ldots\right\}$. But $cl_\omega(\cup_{i\in\mathbb{N}}A_i)=\left\{1,\frac{1}{2},\frac{1}{3},\ldots,0\right\}$. Thus $cl_\omega(\cup_{\lambda\in\Lambda}A_\lambda)\nsubseteq\cup_{\lambda\in\Lambda}cl_\omega(A_\lambda)$ □

2. $\omega-R_i$ −SPACES , FOR $i=0,1$

In this section we introduce some types of weak separation axioms by utilizing the ω −open sets defined in [3]

Definition 2.1: Let $A \subset (X,T)$, then the ω $-$*kernal* of A denoted by $\omega - ker(A)$ is the set
$\omega - ker(A) = \cap\{$O,where O is an ω- open set in (X,T) containing $A\}$.

Proposition 2.2: Let $A \subset (X,T)$, and $x \in X$. Then

$$\omega - ker(A) = \{x \in X : cl_\omega(\{x\}) \cap A \neq \emptyset\}.$$

Proof:

Let A be a subset of X, and $x \in \omega - ker(A)$, such that $cl_\omega(\{x\}) \cap A = \emptyset$. Then $x \notin X \backslash cl_\omega(\{x\})$, which is an ω $-$open set containing A. This contradicts $x \in \omega - ker(A)$. So $cl_\omega(\{x\}) \cap A \neq \emptyset$.
Then Let $x \in X$, be a point satisfied $cl_\omega(\{x\}) \cap A \neq \emptyset$. Assume $x \notin \omega - ker(A)$, then there exists an ω $-$open set G containing A but not x. Let $y \in cl_\omega(\{x\}) \cap A$. Hence G is an ω $-$open set containing y but not x. This contradicts $cl_\omega(\{x\}) \cap A \neq \emptyset$. So $x \in \omega - ker(A)$

Definition 2.3: A topological space (X,T) is said to be *sober* $\omega - R_0$ if $\cap_{x\in X} cl_\omega(\{x\}) = \emptyset$.

Theorem 2.4: A topological space (X,T) is sober $\omega - R_0$ if and only if $\omega - ker(\{x\}) \neq X$ for each $x \in X$.

Proof:

Suppose that (X,T) is sober $\omega - R_0$. Assume there is a point $y \in X$, with $\omega - ker(\{y\}) = X$. Let $x \in X$, then $x \in V$ for any $\omega -$ open set V containing y , so $y \in cl_\omega(\{x\})$ for each $x \in X$. This implies $y \in \cap_{x\in X} cl_\omega x$, which is a contradiction with $\cap_{x\in X} cl_\omega x = \emptyset$.

Now suppose ω $-$kernal$(\{x\}) \neq X$ for every $x \in X$. Assume X is not sober $\omega - R_0$, it mean there is y in X such that $y \in \cap_{x\in X} cl_\omega(\{x\})$, then every $\omega -$ open set containing y must contain every point of X. This implies that X is the unique ω- open set containing y. Therefore ω $-$kernal$(\{y\}) = X$, which is a contradiction with our hypothesis. Hence (X,T) is sober $\omega - R_0$ □

Definition 2.5: A map $f: X \to Y$ is called ω $-$*closed*, if the image of every ω $-$closed subset of X is ω $-$closed in Y.

Proposition 2.6: If X is a space, f is a map defined on X and $A \subseteq X$, then

$$cl_\omega(f(A)) \subseteq f(cl_\omega(A)).$$

Proof:

We have $A \subseteq cl_\omega(A)$, then $f(A) \subseteq f(cl_\omega(A))$
This implies $cl_\omega(f(A)) \subseteq cl_\omega(f(cl_\omega(A))) = f(cl_\omega(A))$.
Hence $cl_\omega(f(A)) \subseteq f(cl_\omega(A)$ □

Theorem 2.7: If $f: X \to Y$ is one to one ω $-$closed map and X is sober $\omega - R_0$, then Y is sober $\omega - R_0$.

Proof:

From Proposition 1.5, we have

$$\cap_{y\in Y} cl_\omega(\{y\}) \subset \cap_{x\in X} cl_\omega(\{f(x)\}) \subset \cap_{x\in X} f(cl_\omega(\{x\}))$$
$$= f(\cap_{x\in X} cl_\omega(\{x\}))$$
$$= f(\emptyset) = \emptyset.$$

Thus Y is sober $\omega - R_0$ □

Definition 2.8: A topological space (X,T) is called $\omega - R_0$ if every ω $-$open set contains the ω $-$closure of each of its singletons.

Theorem 2.9: The topological door space is $\omega - R_0$ if and only if it is $\omega^\star - T_1$.

Proof:

Let x, y are distinct points in X. Since (X,T) is door space so that for each x in , $\{x\}$ is open or closed.
i. 1. When $\{x\}$ is open, hence ω $-$open set in X. Let $V = \{x\}$, then $x \in V$, and $y \notin V$. Therefore since (X,T) is $\omega - R_0$ space, so that $cl_\omega(\{x\}) \subset V$. Then $x \notin X \backslash V$, while $y \in X \backslash V$, where $X \backslash V$ is an ω $-$open subset of X.
2. Whenever $\{x\}$ is closed, hence it is ω $-$closed, $y \in X \backslash \{x\}$, and $X \backslash \{x\}$ is ω $-$open set in X. Then since (X,T) is $\omega - R_0$ space, so that $cl_\omega(\{y\}) \subset X \backslash \{x\}$. Let $V = X \backslash cl_\omega(\{y\})$, then $x \in V$, but $y \notin V$, and V is an ω $-$open set in X. Thus we obtain (X,T) is $\omega^\star - T_1$.
ii. For the other direction assume (X,T) is $\omega^\star - T_1$, and let V be an ω $-$open set of X, and $x \in V$. For each $y \in X \backslash V$, there is an ω $-$open set V_y such that $x \notin V_y$, but $y \in V_y$. So $cl_\omega(\{x\}) \cap V_y = \emptyset$, which is true for each $y \in X \backslash V$. Therefore $cl_\omega(\{x\}) \cap (\cup_{y\in X\backslash V} V_y) = \emptyset$. Then since $y \in V_y$, $X \backslash V \subset \cup_{y\in X\backslash V} V_y$, and $cl_\omega(\{x\}) \subset V$. Hence (X,T) is $\omega - R_0$ □

Definition 2.10: A topological space (X,T) is ω $-$*symmetric* if for x and y in the space X, $x \in cl_\omega(\{y\})$ implies $y \in cl_\omega(\{x\})$.

Proposition 2.11: Let X be a door ω $-$symetric topological space . Then for each $x \in X$, the set $\{x\}$ is ω $-$closed.

Proof:

Let $x \neq y \in X$, since X is a door space so $\{y\}$ is open or closed set in X. When $\{y\}$ is open, so it is ω $-$open, let $V_y = \{y\}$. Whenever $\{y\}$ is ω $-$closed , $x \notin \{y\} = cl_\omega(\{y\})$. Since X is ω $-$symetric we get $y \notin cl_\omega(\{x\})$. Put $V_y = X \backslash cl_\omega(\{x\})$, then $x \notin V_y$ and $y \in V_y$, and V_y is ω $-$open set in X. Hence we get for each $y \in X \backslash \{x\}$ there is an ω $-$open set V_y such that $x \notin V_y$ and $y \in V_y$. Therefore $X \backslash \{x\} = \cup_{y\in X\backslash\{x\}} V_y$ is ω $-$open, and $\{x\}$ is ω $-$closed □

Proposition 2.12: Let (X,T) be $\omega^\star - T_1$ topological space, then it is ω $-$symetric space.

Proof:

Let $x \neq y \in X$. Assume $y \notin cl_\omega(\{x\})$, then since X is $\omega - T_1$ there is an open set U containing x but not y, so $x \notin cl_\omega(\{y\})$. This completes the proof □

Volume 1 No.6, December 2011 ISSN 2224-3577
International Journal of Science and Technology

http://www.ejournalofsciences.org

Theorem 2.13: The topological door space is $\omega-$ symmetric if and only if it is $\omega^{\star}-T_1$.

Proof:

Let (X,T) be a door $\omega-$ symmetric space. Then using Proposition 2.11 for each $\in X$, $\{x\}$ is $\omega-$closed set in X. Then Lemma 1.3, we get that (X,T) is $\omega^{\star}-T_1$. On the other hand, assume (X,T) is $\omega^{\star}-T_1$, then directly by Proposition 2.12. (X,T) is $\omega-$ symmetric space □

Corollary 2.14: Let (X,T) be a topological door space, then the following are equivalent:

1. (X,T) is $\omega-R_0$ space.
2. (X,T) is $\omega^{\star}-T_1$ space.
3. (X,T) is $\omega-$ symmetric space.

Proof:

The proof follows immediately from Theorem 2.9 and Theorem 2.13 □

Corollary 2.15: If (X,T) is a topological door space, then it is $\omega-R_0$ space if and only if for each $x\in X$, the set $\{x\}$ is $\omega-$closed set.

Proof:

We can prove this corollary by using Corollary 2.14 and Lemma 1.3 □

Theorem 2.16: Let (X,T) be a topological space contains at least two points. If X is $\omega-R_0$ space, then it is sober $\omega-R_0$ space.

Proof:

Let x and y are two distinct points in X. Since (X,T) is $\omega-R_0$ space so by Theorem 2.8 it is $\omega^{\star}-T_1$. Then Lemma 1.3 implies $cl_{\omega}(\{x\})=\{x\}$ and $cl_{\omega}(\{y\})=\{y\}$. Therefore $\cap_{p\in X}\ cl_{\omega}(\{p\})\subset cl_{\omega}(\{x\})\cap cl_{\omega}(\{y\})=\{x\}\cap\{y\}=\emptyset$. Hence (X,T) is sober $\omega-R_0$ space □

Definition 2.17: A topological door space (X,T) is said to be $\omega-R_1$ ***space*** if for x and y in , with $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$, there are disjoint $\omega-$open set U and V such that $cl_{\omega}(\{x\})\subset U$, and $cl_{\omega}(\{y\})\subset V$.

Theorem 2.18: The topological door space is $\omega-R_1$ if and only if it is $\omega^{\star}-T_2$ space.

Proof:

Let x and y be two distinct points in X. Since X is door space so
for each x in X, The set $\{x\}$ is open or closed.
i. If $\{x\}$ is open. Since $\{x\}\cap\{y\}=$ □, then $\{x\}\cap cl_{\omega}\{y\}=$ □. Thus $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$.
ii. Whenever $\{x\}$ is closed, so it is $\omega-$closed and $cl_{\omega}\{x\}\cap\{y\}=\{x\}\cap\{y\}=$ □. Therefore $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$. We have (X,T) is $\omega-R_1$ space, so that there are disjoint $\omega-$open sets U and V such that $x\in cl_{\omega}(\{x\})\subset U$, and $y\in cl_{\omega}(\{y\})\subset V$, so X is $\omega^{\star}-T_2$ space.

For the opposite side let x and y be any points in X, with $cl_{\omega}(\{x\})\neq cl_{\omega}(\{y\})$. Since every $\omega^{\star}-T_2$ space is $\omega^{\star}-T_1$ space so by (3) of Theorem 2.2.15 $cl_{\omega}(\{x\})=\{x\}$ and $cl_{\omega}(\{y\})=\{y\}$, this implies $x\neq y$. Since X is $\omega^{\star}-T_2$ there are two disjoint $\omega-$open sets U and V such that $cl_{\omega}(\{x\})=\{x\}\subset U$, and $cl_{\omega}(\{y\})=\{y\}\subset V$. This proves X is $\omega-R_1$ space □

Corollary 2.19: Let (X,T) be a topological door space. Then if X is $\omega-R_1$ space then it is $\omega-R_0$ space.

Proof:
Let X be an $\omega-R_1$ door space. Then by Theorem 2.17 X is $\omega^{\star}-R_2$ space. Then since every $\omega^{\star}-T_2$ space is $\omega^{\star}-T_1$, so that by Theorem 2.9, X is $\omega-R_0$ space.

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