Estimates for Comonotone Polynomial Approximation in L_p

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Abstract

When we approximate a function f in $L_p, \infty > p \geq 1$ which changes its monotonicity finitely many, say s time, in $[-1,1]$, we wish some times that the approximating polynomial follow these changes of monotonicity. However it is well known that this requirement restricts very much the degree of approximation that the polynomials can achieve, namely the rate of $\omega_2\left(f,n^{-1}\right)_p$. In [1] we prove that relaxing the comonotonicity requirements in very small intervals about the interior extremes and near the end points, what we called nearly comonotone approximation allows the polynomials to achieve a pointwise approximation rate of $\omega_3\left(f,n^{-1}\right)_p$. Also we proved that even when we relax the requirement of monotonicity of the polynomials on sets of measures approaching 0, ω_4 is not reachable. We prove here that when f belongs to the Sobolev space, allow estimates involving the kth modulus of smoothness of f' for $k \geq 1$.

1. Introduction

Let $f \in L_p[-1,1]$, change monotonicity $s \geq 1$ times on $I = [-1,1]$. Let $Y_s = \{y_i\}_{i=1}^s$ set of points so that $-1 < y_s < \ldots < y_1 < 1$. Denote by $\Delta^{(1)}(Y_s)$ the collection of functions which change monotonicity at the points y_i. A polynomial $p_n \in \mathrm{P}_n$ the space of polynomials of degree not exceeding n, is said to be *comonotone* with f, on a set $E \subset I = [-1,1]$ if and only if

$$p_n'(x)\prod_{i=1}^{s}(x-y_i) \geq 0 \qquad \forall x \in E .$$

The *degree of approximation* of $f \in \Delta^{(1)}(Y_s) \cap L_p[-1,1]$ by comonotone polynomials is measured in the L_p norm for $1 \leq p < \infty$, and defined by

$$E_n^1(f,Y_s)_p := \inf_{p_n \in \mathrm{P}_n \cap \Delta^{(1)}(Y_s)} \|f - p_n\|_p .$$

We also denote by $W_p^k[a,b]$, the set of all functions f on $[a,b]$ such that $f^{(k-1)}$ is absolutely continuous, and $f^{(k)} \in L_p$, such a space is called the *Sobolev* space. The *rth symmetric difference* of f is given by

$$\Delta_h^r(f,x,[a,b]) := \Delta_h^r(f,x) := \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh}{2} + ih\right), & x \pm \frac{rh}{2} \in [a,b] \\ 0, & o.w. \end{cases}$$

Then the *rth usual modulus of smoothness* of $f \in L_p[a,b]$ is defined by

$$\omega_r(f,\delta,[a,b])_p := \sup_{0<h\leq\delta} \left\| \Delta_h^r(f,.) \right\|_{L_p[a,b]}, \delta \geq 0. \text{ [2]}$$

We will also use the so called $\tau - mdulus$ (or Sendov-Popov modulus), an averaged modulus of smoothness, defined for bounded measurable functions on $[a,b]$ by

$$\tau_r(f,\delta,[a,b])_p := \left\| \omega_r(f,.,\delta) \right\|_{L_p[a,b]}, \text{ [5]}$$

where

$$\omega_r(f,x,\delta) := \sup\left\{ \left| \Delta_h^r(f,y) \right| : y \pm \frac{rh}{2} \in \left[x - \frac{r\delta}{2}, x + \frac{r\delta}{2} \right] \cap [a,b] \right\}$$

is the *rth local modulus of smoothness* of f . From the definition one can easily see

$$\tau_r(f,\delta,[a,b])_\infty := \omega_r(f,\delta,[a,b])_\infty .$$

The following relationship between the ω and τ moduli holds for any $f \in W_p^1[a,b], 1 \leq p \leq \infty$.

$$\tau_r(f,\delta,[a,b])_p \leq c(r)\delta\omega_{r-1}(f',\delta,[a,b])_p . \text{ [6]} \qquad (1.1)$$

If the interval $I = [-1,1]$, is used in any of the above notation it will be omitted for the sack of simplicity, for example:

$$\omega_r(f,\delta)_p := \omega_r(f,\delta,[-1,1])_p ,$$

and we will also denote

$$\omega_r(f,\delta) := \omega_r(f,\delta,[-1,1])_\infty .$$

The moduli ω and τ, measure the smoothness of f over the interval uniformly. It is well known that polynomials approximate better near the end points of the interval than in the middle, and this leads to either pointwise estimates (if $p=\infty$) or the introduction of nonuniform moduli of smoothness the non uniform modulus that we use is the rth Ditizian Totik [3] modulus of smoothness defined for $f \in L_p(I)$

$$\omega_r^{\varphi}(f,\delta,I)_p := \sup_{0<h\le\delta} \left\|\Delta_{h\varphi(\cdot)}^r(f,.)\right\|_{L_p(I)},$$

with $\varphi(x)=\sqrt{1-x^2}$. We have

$$\omega_r^{\varphi}(f,\delta)_p \le \omega_r(f,\delta)_p \le \tau_r(f,\delta)_p \le 2^{\frac{1}{p}}\omega_r(f,\delta)_\infty, \quad 1\le p\le\infty .[6]$$

However In Lemma 2.2.5 in [1] we proved that the moduli ω_r and ω_r^{φ} for an f defined on $J=[a,b]\subset[-1,1]$ are equivalent if $|J|\approx|\Delta_n(a)|$ with $\Delta_n(a)=n^{-1}\sqrt{1-a^2}+n^{-2}$:

Lemma 1.1 *Let $[a,b]\subset[-1,1]$ be such that $(b-a)\le\rho_n(a)$, where $c_1\ge 1$ is an absolute constant. Then for any nonnegative integer r there is a constant $c(r)$ such that*

$$\omega_r^{\varphi}\left(f,n^{-1},[a,b]\right)_p \ge c(r)\omega_r(f,\rho_n(a),[a,b])_p .$$

Now let us turn to the comonotone approximation

In [1](Corollary 2.1.4, p52) we proved that

Theorem 1.2 *If $f\in L_p(I)\cap\Delta^{(1)}(Y_s)$, then there is a constant $A(s)$ such that for $n>\frac{A(s)}{d(Y_s)}$, there is a polynomial $p_n\in \mathrm{P}_n\bigcap\Delta^{(1)}(Y_s)$ satisfies*

$$\|f-p_n\|_p \le c(s)\omega_2^{\varphi}\left(f,n^{-1}\right)_p \le c(s)\tau_2\left(f,n^{-1}\right)_p$$

where $d(Y_s)=\min\{1+y_1,y_2,...,y_s-y_{s-1},1-y_s\}$. The constant $c(s)$ depends only on s. On the other hand one cannot replace ω_2^{φ} in theorem 1.2 by ω_3, where ω_k denotes the modulus of smoothness of order *k*. It is quite natural to ask whether one can strengthen theorem 1.2 in the sense of being able to replace ω_2 by moduli of smoothness of higher order , if one willing to allow p_n not to be comonotone with *f* on a rather small subset of I, what we called nearly comonotone approximation. In theorem 3.1.2 p.72 , in [1] we proved that in such a case it is possible to achieve the estimates

$$\|f - p_n\|_p \le c(s)\tau_3\left(f, n^{-1}\right)_p.$$

However this improvement can not be extend to ω_4 or τ_4. In theorem 4.1.1 p. 87 [1] we show that even when we relax the requirement of monotonicity of the polynomials on sets of measures approaching zero, ω_4 or τ_4 is not reachable.

We let

$$O(h,Y_s) = I \cap \bigcup_{i=1}^{s} \left(y_i - \Delta_n(y_i), y_i + \Delta_n(y_i)\right)$$

In this paper we prove that if we assume f belongs to the Sobolev space $W_p^1(I)$., then we can obtain estimates involving moduli of higher orders

Theorem 1.3. *For each* $A > 0$, *there is a constant* $c(k,s,A)c$ *for which if* $f \in \Delta^{(1)}(Y_s) \cap W_p^1(I)$ *then for every* $n \ge k$ *a polynomial* $p_n \in \mathrm{P}_n$ *which is comonotone with f on* $I \setminus \mathrm{O}(A/n, Y_s)$ *exists such that*

$$\|f - p_n\|_p \le c(k,s,A)\frac{1}{n}\left(\omega_k^{\varphi}\left(f', n^{-1}\right)_p + \tau_k\left(f', n^{-1}\right)_p\right). \qquad (1.2)$$

2. The proof of the main result

Let $k \ge 1$ be fixed. We construct a piecewise polynomial

$$S \in \Sigma_{k+1, O(Y_s, n)}, \qquad (2.1)$$

which is comonotone with *f* on $I \setminus O(Y_s, nm)$, and sufficiently close to it. To this end we introduce the following

Lemma 2.1 [1] *If f is monotone function in* $W_p^1[a,b], h < 1$, *the there is a monotone polynomial* $p_{r-1} \in P_{r-1}$ *interpolating f at 0 and h, such that*

$$\|f - p_{r-1}\|_{L_p[0,h]} \le ch\omega_{r-1}(f', h, [0,h])_p.$$

For $I_{j,n} \not\subset O(Y_s, n)$, there is a polynomial $p_j = p_{j,n}$ of degree $\le k$, which interpolates *f* at both end points of $I_{j,n}$ for which

$$\|f - p_j\|_{L_p[I_{j,n}]} \le c|I_{j,n}|\omega_k\left(f', |I_{j,n}|, |I_{j,n}|\right)_p \qquad (2.2)$$

Then from Theorem 1.2 since $\omega_k\left(f', |I_{j,n}|, |I_{j,n}|\right)_p \le c\omega_k^{\varphi}\left(f', n^{-1}\right)_p$, so we have

$$\|f - p_j\|_{L_p[I_{j,n}]} \le cn^{-1}\omega_k^{\varphi}\left(f', n^{-1}\right)_p \qquad (2.3)$$

Here and for the rest of the proof constants *c* are absolute constants and may depend on *k* and *s* or *k*, and *m*, they may differ at occurrences, even if they appear in the same line.
Then
Let

$$I_j = \left[a + \frac{(j-1)(b-a)}{m}, a + \frac{j(b-a)}{m} \right], \ j = 1,2,\ldots,m,$$

and set

$$\hat{O}_i = \hat{O}_{i,m}(Y_s) = \left[a + \frac{(j-2)(b-a)}{m}, a + \frac{(j+1)(b-a)}{m} \right] \bigcap [a,b].$$

If

$$\left[a + \frac{(j-1)(b-a)}{m}, a + \frac{j(b-a)}{m} \right] \ y_i \in 1 \le i \le s, \text{ and } \hat{O} = \bigcup_{i=1}^{s} \hat{O}_i .$$

For each $\hat{O}_v$, we obtain by Lemma 2.1 , a polynomial q_v of degree $\le k$ which is comonotone with f on $\hat{O}_v / O(Y_s, nm)$, such that

$$\| f - q_v \|_{L_p(\hat{O}_v)} \le c |\hat{O}_v| \omega_k \left(f', |\hat{O}_v|, \hat{O}_v \right)_p \le \frac{c}{n} \omega_k^{\varphi} \left(f', \frac{1}{n} \right)_p . \tag{2.4}$$

The piecewise polynomial

$$\hat{S}(x) = \begin{cases} q_v(x) & if \quad x \in \hat{O}_v, \\ p_j(x) & if \quad x \in I_{j,n} \setminus O \end{cases},$$

has discontinuities at the end points of $\hat{O}_v$, and we alter it to obtain continuous piecewise *S* which is comonotone with *f* on $I \setminus O(Y_s, nm)$, satisfies (2.1), and by virtue of (2.3) and (2.4)

$$\| f - S \|_p \le \frac{c_4}{n} \omega_k^{\varphi} \left(f', \frac{1}{n} \right)_p . \tag{2.5}$$

Since *S* is a polynomial of degree at most *k* on each $\hat{O}_v$, then it changes monotonicity at most *k-1* there. Hence there is $Y_* \subset O(Y_s, nm)$, containing $s_* \le (k-1)s$ points such that either $S \in \Delta^{(1)}(Y_*)$, or $-S \in \Delta^{(1)}(Y_*)$. Since $m \ge 2$, it follows that $O(Y_s, 2n) \subset O(Y_s, n)$, whence

$$S \in \Sigma_{k+1, O(Y_*, 2n)} .$$

In our proof of this theorem we make use of the following lemma 3.2.22 , p. 81 from [1]

Lemma 2.2. *If* $S \in \Sigma_{k1, O(Y_*, n)} \bigcap \Delta^{(1)}(Y_s)$, *then*

$$E_{c_1 n}^{(1)}(S, Y_s) \le c(k,s) \tau_m (S, n^{-1})_p .$$

In particular

$$E_{c_1 n}^{(1)}(S, Y_s) \le c(k,s) n^{-1} \omega_{r-1}(f', n^{-1})_p,$$

where $c_1 = c_1(k,s)$.

Then applying Lemma 2.2, and conclude that there is a polynomial *p* of degree $\le c_7 n$, which is comonotone with *S* everywhere on *I*, and such that

$$\|S - P\|_p \le c_8 \tau_{k+1}\left(S, \frac{1}{2n}\right)_p. \qquad (2.6)$$

It is readily seen that *p* is comonotone with *f* on $I \setminus O(Y_s, nm)$, and combining (2.5) and (2.6) we have

$$\begin{aligned}\|f - p\|_p &\le \|f - S\|_p + \|S - p\|_p \\ &\le \|f - S\|_p + c_8 \tau_{k+1}\left(S, \frac{1}{2n}\right)_p \\ &\le \|f - S\|_p + c_8 \tau_{k+1}\left(S - f, \frac{1}{2n}\right)_p + c_8 \tau_{k+1}\left(f, \frac{1}{2n}\right)_p \\ &\le c n^{-1} \omega_k^{\varphi}(f', n^{-1})_p + c_8 \tau_{k+1}\left(f, \frac{1}{2n}\right)_p.\end{aligned}$$

Then by the relationship (1.1), we have

$$\|f - p\|_p \le c(r) \frac{1}{n} \left(\omega_k^{\varphi}(f', n^{-1})_p + \omega_k(f, n^{-1})_p\right).$$

Given $n \ge c_7$, and $m \ge 2$. Applying the above to $n_1 = [n / c_7]$ and $m_1 \ge m c_7$ so big that nm divides $n_1 m_1$. We obtain a polynomial p_n of degree $\le n$, which is comonotone with *f* on $I \setminus O(Y_s, nm)$ such that (1.2) satisfied. Thus a proper choice of $m = m(A)$ yields our theorem for $n \ge c_7$ and for $k \le n \le c_7$, our theorem follows from Lemma 2.1. This completes the proof ♠

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