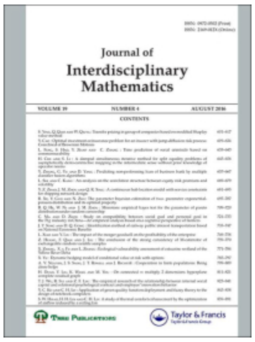

Taylor & Francis
Taylor & Francis Group

Journal of Interdisciplinary Mathematics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/tjim20

On the n normed space of L_∞ measurable functions

Rehab Amer Kamel, Mayada Ali Kareem & Ahmed Hadi Hussain

To cite this article: Rehab Amer Kamel, Mayada Ali Kareem & Ahmed Hadi Hussain (2022): On the n normed space of L_∞ measurable functions, Journal of Interdisciplinary Mathematics, DOI: 10.1080/09720502.2022.2052569

To link to this article: https://doi.org/10.1080/09720502.2022.2052569

Published online: 04 Jul 2022.

Journal of Interdisciplinary Mathematics
ISSN: 0972-0502 (Print), ISSN: 2169-012X (Online)
DOI : 10.1080/09720502.2022.2052569

Taylor & Francis
Taylor & Francis Group

On the n normed space of L_∞ measurable functions

Rehab Amer Kamel [§]
Mayada Ali Kareem [†]
Department of Mathematics
College of Education for Pure Sciences
University of Babylon
Iraq

Ahmed Hadi Hussain*
Department of Energy Engineering
College of Engineering Al-Musayab
University of Babylon
Iraq

Abstract

In this paper we study the space $L_\infty(X)$ all functions that are essentially bounded, since X is a measureable space for n disjoin subsets of X, can be viewed together an n-norm space $(X, \|.,\ldots,.\|_\infty)$ normed space. We view several features of this n normed space like as a derived norm, property of completeness, fixed-point-theorem.

Subject Classification: *41A50, 30E10.*

Keywords: *n normed space, Derived norm, Measurable functions space L_∞, Equivalent two norms, Completeness, Contractive mapping-theorem.*

1. Introduction

Definition 1.1 [6] : Let $\boldsymbol{n}$ be a positive integer and $\boldsymbol{X}$ is a real vector space of dimension bigger than $\boldsymbol{n}$. A mapping $\|.,\ldots,.\| : \boldsymbol{X}^n \to \boldsymbol{R}$ with the following four properties:

[§] *E-mail:* pure.rehab.amer@uobabylon.edu.iq
[†] *E-mail:* pure.meyada.ali@uobabylon.edu.iq
* *E-mail:* met.ahmed.hadi@uobabylon.edu.iq (Corresponding Author)

1. $\|x_1, \ldots, x_n\| = 0 \leftrightarrow x_1, x_2, \ldots.., x_n$ are dependent oflinearly;
2. $\|x_1, \ldots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, \ldots, x_n\| = |\alpha| \|x_1, \ldots, x_n\|$ for any $\alpha \in R$;
4. $\|x + x', x_2, \ldots, x_n\| \le \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$,

then the above properties with $(X, \|., \ldots, .\|)$ are called an n normed space.

Which inner product of real vector space $(X, \langle .,. \rangle)$ given with the standard n norm $\|x_1, \ldots, x_n\| := (\det(\langle x_i, x_j \rangle))^{\frac{1}{2}}$. The above definition represents also the volume of the n dimensional parallelepiped spread by $x_1, x_2, \ldots, x_n$ in X. On R^n, this n norm knows as $\|x_1, \ldots, x_n\| := |\det(x_{ij})|$, where $x_i = (x_{i1}, \ldots, x_{in}) \in R^n, i = 1, \ldots, n$. The concept of 2 normed spaces was improved by Gahler [4] since middle 1960. The theory of n normed spaces was studied by many researchers, firstly by Misiak[6] (also see the sources [2], [3],[4], [5]). Through this work, we'll discuss the space $L_\infty(X)$ the space of modulus equivalence almost everywhere (equivalence classes) of functions like as $\sup_{x \in X} |f(x)| < \infty$, and its normal parameters n, which is known as a generalization of the standard norm $\|f_\infty\| := \sup_{x \in X} |f(x)|$. From [5] we take the definition $\|., \ldots, .\|_\infty$ on $L_\infty(X) \times L_\infty(X) \times \ldots \times L_\infty(X)$ (n factors), $\|f_1, \ldots, f_n\|_\infty := \sup_{x_1 \in X} \ldots \sup_{x_n \in X} |\det(f_i(x_j))|$.

Theorem 1.2 : *Let* $f, f^*, f_2, \ldots, f_n \in L_\infty(X) \times L_\infty(X) \times \ldots \times L_\infty(X)$ (n *factors*), *then* $\|f + f^*, f_2, \ldots, f_n\|_\infty \le (\|f, f_2, \ldots, f_n\|_\infty + \|f^*, f_2, \ldots, f_n\|_\infty)$.

Theorem 1.3 [7] : *Suppose that* $p_1, \ldots.., p_m > 0$ *real numbers such that* $\sum_{\alpha=1m}^{m} 1/p_{\acute{a}} = 1$ *and* f_α *be measurable functions for* $\alpha = 1, \ldots., m$. *Then* $\prod_{\alpha=1}^{m} f_\alpha \in L^1$ *and* $\int \prod_{\alpha=1}^{m} |f_\alpha| \, d\mu \le \prod_{\alpha=1}^{m} \|f_\alpha\|_{p_\alpha}$.

Auxiliary theorem 1.4 [1] : *Suppose that X be a vector space of finite-dimensional, then* $\|.\|$ *and* $\|.\|^*$ *are equivalent if* $\forall x \in X \exists$ *constants* $0 < \mathcal{B}, \aleph < \infty \ni B$ $\|x^*\| \le \|x\| \le \aleph \|x^*\|$.

2. The Main Results

Theorem 2.1 : *The inequality* $\|f_1, \ldots, f_n\|_\infty \le (n^2!) \; (\|f_1\|_\infty \|f_2\|_\infty \cdots \|f_n\|_\infty)$, *holds whenever* $f_1, f_2, \ldots, f_n \in L_\infty(X)$.

***Proof*:** Suppose Θ be a set of all permutations of $\{1,2,\ldots,n\}$. By Minkowski's and triangle inequalities, we have

$$\begin{aligned}\|f_1,\ldots,f_n\|_\infty &:= \sup_{x_1\in X}\ldots\sup_{x_n\in X}|\det(f_i(x_j))| \\ &= \sup_{x_1\in X}\ldots\sup_{x_n\in X}\left|\sum_{\vartheta=\{j_1,\ldots,j_n\}\in\Theta}\operatorname{sgn}(\vartheta)(f_1(x_{j1})f_2(x_{j2})\ldots f_n(x_{jn}))\right| \\ &\le (n!)\sup_{x_1\in X}\ldots\sup_{x_n\in X}\left(\sum_{\vartheta=\{j_1,\ldots,j_n\}\in\Theta}|f_1(x_{j1})f_2(x_{j2})\ldots f_n(x_{jn})|\right) \\ &\le (n!)\sum_{\vartheta=\{j_1,\ldots,j_n\}\in\Theta}\left(\sup_{x_1\in X}|f_1(x_{j1})|\ldots\sup_{x_n\in X}|f_n(x_{jn})|\right)\end{aligned}$$

By Theorem 1.3, we get

$$\begin{aligned}\|f_1,\ldots,f_n\|_\infty &= (n!)\sum_{\vartheta=\{(j_1,\ldots,j_n\}\in\Theta}(\|f_1\|_\infty\|f_2\|_\infty\cdots\|f_n\|_\infty) \\ &\le (n^2!)(\|f_1\|_\infty\|f_2\|_\infty\cdots\|f_n\|_\infty).\end{aligned}$$

Theorem 2.2 : $\|f_1,\ldots,f_n\|_\infty$ *is an* n *normed space for* L_∞.

***Proof* :** We necessity to proof which $\|f_1,\ldots,f_n\|_\infty$ corresponds the three conditions of an n norm 2. , 3. are obvious, we'll prove the first condition.

Assume that $\|f_1,\ldots,f_n\|_\infty = 0$. Then $\det\begin{bmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n)\end{bmatrix} = 0$ For all i and, that leads to $f_1, f_2, \ldots, f_n$ are dependent of linearly, the converse side is obvious. Case 4. satisfies by using Theorem 1.2.

Corollary 2.3 : $(L_\infty, \|f_1,\ldots,f_n\|_\infty)$ *it defines an* n *normed space*.

3. Completeness

Definition of converge sequence.

Suppose that a sequence of functions $\mathcal{F}_k(x)\in n$ normed space $(X,\|.,\ldots,.\|)$ is said to be converge to some $\mathcal{F}\in X$ in the n norm whenever, $\lim_{k\to\infty}\|\mathcal{F}_k(x)-\mathcal{F}, f_2,\ldots,f_n\| = 0$, for every $f_2,\ldots,f_n\in X$. Also a sequence of functions $\mathcal{F}_k(x)$ is called Cauchy sequence of functions in the n normed space if, $\lim_{m,k\to\infty}\|\mathcal{F}_k(x)-\mathcal{F}_m(x), f_2,\ldots,f_n\| = 0$, for every $f_2,\ldots,f_n\in X$. If every Cauchy sequence in X converges to some $\mathcal{F}\in X$, then X is called a complete n normed space.

Auxiliary theorem 3.1 : *A sequence of functions in the* n *normed space* $(L_\infty, \|.,\dots,.\|_\infty)$ *is convergent if f it is convergent in the standard normed space* $(L_\infty, \|.\|_\infty)$. *likewise, a sequence of functions in the* n *normed space* $(L_\infty, \|.,\dots,.\|_\infty)$ *be Cauchy if f Cauchy in the standard normed space* $(L_\infty, \|.\|_\infty)$. *The one parts of auxiliary theorem* 3.2 *happens from Theorem* 2.1. *To solve the second parts we are using a derived norm, defined with reference to* $\{\chi A_1,\dots,\chi A_n\}$, *where* $A_1,\dots,A_n$ *be disjoint sets of non-negative measureable and* $\chi A : X \to R$ *stands for the characteristic function of a set* $A \subset X$, *i.e.,* $\chi A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$. *So, we can define a derived norm as follows*:

$$\|f\|_\infty^* := \sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|f, \chi A_{i2},\dots,\chi A_{in}\|_\infty .$$

Theorem 3.2 : *The derived norm* $\|.\|_\infty^*$ *is analogous to the standard norm* $\|.\|_\infty$. *Properly, we have*

$$\frac{n\|\chi A_1,\dots,\chi A_n\|_\infty}{(2n-1)(\|\chi A_1\|_\infty+\dots+\|\chi A_n\|_\infty)}\|f\|_\infty \le \|f\|_\infty^*$$

$$\le (n^2!)\left(\sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|\chi A_{i2}\|_\infty \dots\dots \|\chi A_{in}\|_\infty\right)\|f\|_\infty$$

Proof **:** For every $f \in L_\infty$ and any subset $\{i_2,\dots.,i_n\} \subset \{1,\dots,n\}$, we compute

$$\|f, \chi A_{i2}, \chi A_{i3},\dots\dots,\chi A_{in}\|_\infty \le n^2!\|f\|_\infty\|\chi A_{i2}\|_\infty \dots\|\chi A_{in}\|_\infty$$

By Theorem 2.1 , we get

$$\|f\|_\infty^* = \sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|f, \chi A_{i2},\dots,\chi A_{in}\|_\infty$$
$$\le (n^2!)\left(\sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|\chi A_{i2}\|_\infty \dots\|\chi A_{in}\|_\infty\right)\|f\|_\infty .$$

To prove the reverse inequality, we obtain

$$\|f\|_\infty\|\chi A_1,\dots,\chi A_n\|_\infty = \sup_{x_1\in X}\dots\sup_{x_n\in X}\left|f(x)\begin{vmatrix}\chi A_1(x_1) & \cdots & \chi A_n(x_1)\\ \vdots & \ddots & \vdots \\ \chi A_1(x_n) & \cdots & \chi A_n(x_n)\end{vmatrix}\right|$$

By Minkowski's inequality, we have

$$\sup_{x_1\in X}\ldots\sup_{x_n\in X}\left| f(x)\begin{vmatrix}\chi A_1(x_1) & \cdots & \chi A_n(x_1)\\ \vdots & \ddots & \vdots\\ \chi A_1(x_n) & \cdots & \chi A_n(x_n)\end{vmatrix}\right|$$

$$\le \sup_{x_1\in X}\ldots\sup_{x_n\in X}\left| \chi A_1(x_1)\begin{vmatrix} f(x) & \cdots & \chi A_n(x)\\ \vdots & \ddots & \vdots\\ f(x_n) & \cdots & \chi A_n(x_n)\end{vmatrix}\right| +\ldots$$

$$+\sup_{x_1\in X}\ldots\sup_{x_n\in X}\left| \chi A_1(x_n)\begin{vmatrix} f(x_1) & \cdots & \chi A_n(x_1)\\ \vdots & \ddots & \vdots\\ f(x) & \cdots & \chi A_n(x)\end{vmatrix}\right|$$

$$+\sup_{x_1\in X}\ldots\sup_{x_n\in X}\left| \chi A_2(x)\begin{vmatrix}\chi A_1(x_1) & f(x_1)\cdots & \chi A_n(x_1)\\ \vdots & \ddots & \vdots\\ \chi A_1(x_n) & f(x_n)\cdots & \chi A_n(x_n)\end{vmatrix}\right| +\ldots$$

$$+\sup_{x_1\in X}\ldots\sup_{x_n\in X}\left| \chi A_n(x)\begin{vmatrix}\chi A_1(x_1) & \cdots & f(x_1)\\ \vdots & \ddots & \vdots\\ \chi A_1(x_n) & \cdots & f(x_n)\end{vmatrix}\right|$$

$$= n\left\|\chi A_1\right\|_\infty \left\|f,\chi A_2,\ldots,\chi A_n\right\|_\infty + \left\|\chi A_2\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_n\right\|_\infty$$

$$+\ldots+\left\|\chi A_n\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_{n-1}\right\|_\infty .$$

Consequently, we get

$$\left\|f\right\|_\infty \left\|\chi A_1,\chi A_2,\ldots,\chi A_n\right\|_\infty$$
$$\le n\left\|\chi A_1\right\|_\infty \left\|f,\chi A_2,\ldots,\chi A_n\right\|_\infty + \left\|\chi A_2\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_n\right\|_\infty$$
$$+\ldots+\left\|\chi A_n\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_{n-1}\right\|_\infty$$

$$\left\|f\right\|_\infty \left\|\chi A_2,\chi A_1,\ldots,\chi A_n\right\|_\infty$$
$$\le n\left\|\chi A_2\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_n\right\|_\infty + \left\|\chi A_1\right\|_\infty \left\|f,\chi A_2,\ldots,\chi A_n\right\|_\infty$$
$$+\ldots+\left\|\chi A_n\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_{n-1}\right\|_\infty$$

$$\left\|f\right\|_\infty \left\|\chi A_n,\chi A_1,\ldots,\chi A_{n-1}\right\|_\infty \le n\left\|\chi A_n\right\|_\infty \left\|f,\chi A_1,\ldots,\chi A_{n-1}\right\|_\infty +$$
$$\left\|\chi A_1\right\|_\infty \left\|f,\chi A_n,\ldots,\chi A_{n-1}\right\|_\infty +\ldots+\left\|\chi A_{n-1}\right\|_\infty \left\|f,\chi A_n,\ldots,\chi A_{n-2}\right\|_\infty ,$$

therefore, see that and by auxiliary theorem 1.4, we get

$$n\|f\|_\infty \|\chi A_1,\chi A_2,\dots,\chi A_n\|_\infty \le (2n-1)\|\chi A_1\|_\infty \|f,\chi A_2,\dots,\chi A_n\|_\infty + \dots +$$

$$(2n-1)\|\chi A_n\|_\infty \|f,\chi A_1,\dots,\chi A_{n-1}\|_\infty \Rightarrow$$

$$\|f,\chi A_2,\dots,\chi A_n\|_\infty \le \sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|f,\chi A_{i2},\dots,\chi A_{in}\|_\infty = \|f\|_\infty^*$$

$$\|f,\chi A_2,\dots,\chi A_{n-1}\|_\infty \le \sup_{\{i_2,\dots,i_n\}\subset\{1,\dots,n\}} \|f,\chi A_{i2},\dots,\chi A_{in}\|_\infty = \|f\|_\infty^*, \text{ so we get}$$

$$n\|f\|_\infty \|\chi A_1,\chi A_2,\dots,\chi A_n\|_\infty \le (2n-1)(\|\chi A_1\|_\infty + \|\chi A_2\|_\infty + \dots + \|\chi A_n\|_\infty)$$

$$\|f\|_\infty^*.$$

Remark 3.3 : Theorem 3.2 help us in special that $\|.\|_\infty$ is controlled by $\|.\|_\infty^*$. So thus we in fact necessity to solve Auxiliary theorem 3.1.

Solution of auxiliary theorem 3.2 : *Suppose that a sequence of functions* $\mathcal{F}_k(x)$ *converges to* $\mathcal{F} \in L_\infty$ *in* n *norm* $\|.,\dots,.\|_\infty$. *With respect to* $\{\chi A_1,\dots,\chi A_n\}$, *define* $\|.\|_\infty^*$ *as before. Then, since*

$$\lim_{k\to\infty} \|\mathcal{F}_k(x)-\mathcal{F},\chi A_2,\chi A_3,\dots,\chi A_n\|_\infty = 0,$$
$$\lim_{k\to\infty} \|\mathcal{F}_k(x)-\mathcal{F},\chi A_1,\chi A_3,\dots,\chi A_n\|_\infty = 0,$$

and $\lim_{k\to\infty} \|\mathcal{F}_k(x)-\mathcal{F},\chi A_1,\chi A_2,\dots,\chi A_{n-1}\|_\infty = 0$, *we have* $\lim_{\to\infty} \|\mathcal{F}_k(x)-\mathcal{F}\|_\infty^*$ $=0$, *that is a sequence of functions* $\mathcal{F}_k(x)$ *in* $\|.\|_\infty^*$ *converges to several* $\mathcal{F} \in L_\infty$. *A norm* $\|.\|_\infty$ *is grabbed by the norm* $\|.\|_\infty^*$, *and so we conclude that* $\mathcal{F}_k(x)$ *also converse to* $\mathcal{F}$ *in* $\|.\|_\infty^*$. *Theorem* 2.1. *shows the second side.*

Theorem 3.4 : *The* n *normed space* $(L_\infty, \|.,\dots,.\|_\infty)$ *is a complete space.*

Proof : Suppose a $\mathcal{F}_k(x)$ be Cauchy sequence of functions $\in L_\infty$ in the n norm $\|.,\dots,.\|_\infty$. consequently, by Auxiliary theorem 3.1 a sequence $\mathcal{F}_k(x)$ is Cauchy sequence of functions with reference to the standard norm $\|.\|_\infty$. The space L_∞ is complete in the norm $\|.\|_\infty$, so that a sequence of functions $\mathcal{F}_k(x)$ converges to several $\mathcal{F} \in L_\infty$ in the norm $\|.\|_\infty$. By the second part of

Auxiliary theorem 3.1, a sequence of functions $\mathcal{F}_k(x)$ converges to $\mathcal{F}$ in $\|.,\ldots,.\|_\infty$. Hence the n normed space $(L_\infty, \|.,\ldots,.\|_\infty)$ is a complete space.

Corollary 3.5: *The space* $(L_\infty(X), \|.\|_\infty^*)$ *is complete Banach space. We'll employ "derived norm" to show the following "contractive mapping" in the* n *normed space* $(L_\infty \|.,...,.\|_\infty)$.

Theorem 3.6 (Contractive Mapping) : *Let S be a mapping* $S : L_\infty \to L_\infty$ *which is contractive with respect to characteristic function* $\{\chi A_1, \ldots, \chi A_n\}$ *in* $L_\infty(X)$, *and some constant* $C \in (0,1)$ *such that the inequality*

$$\|Sf - Sf', \chi A_{i2}, \ldots, \chi A_{in}\|_\infty \leq C\|f - f', \chi A_{i2}, \ldots, \chi A_{in}\|_\infty$$

Satisfy for all $f, f',$ *in* L_∞ *and* $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. *Then the mapping S has only one fixed point belong to* L_∞.

Proof **:** For every $f, f' \in L_\infty$, so we get

$$\|Sf - Sf'\|_\infty^* = \sup_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|Sf - Sf', \chi A_{i2}, \ldots, \chi A_{in}\|_\infty$$

$$\leq C\left(\sup_{\{i_2,\ldots,i_n\}\subset\{1,\ldots,n\}} \|Sf - Sf', \chi A_{i2}, \ldots, \chi A_{in}\|_\infty\right) = C\|f - f'\|_\infty^*.$$

and by Corollary 3.5 $\Rightarrow S$ is contractive depend to $\|.\|_\infty^*$.

$\Rightarrow S$ contained one fixed point belong to L_∞.

4. Conclusion

We can communicate the several properties of n-norm space of L_∞ to derived norm valent to its standard norm of L_∞ such as convergence and fixed-point theorems.

References

[1] N.L. Carothers, A short course on approximation Theory, Bowling Green State University, U.S.A., (1998).

[2] S. Ekariani, H. Gunawn, J. Lindiarni, On the n-normed space of p-integrable functions, Mathematica Aeterna, Vol. 5, no. 1, 11-19,(2015).

[3] S. Gahler, 2-metrische Raume und ihretopologischeStruktur Math. Nachr. 26, 115-148, (1963)

[4] S.Gahler,Lineare 2- normietreRaume, .Math .Nachr.28,1-43, (1965).

[5] H.Gunawan, The space of p- summable sequences and its natural n-norm, Bull. Austral. Math.Soc.64,137-147, (2001).

[6] A.Misiak, n-inner product spaces,Math. Nachr. 140, 299-319, (1989).

[7] Wing-Sum Cheung, Generalizations of Holders inequality, IJMMS., 26:1, 7-10, (2001).

[8] Xujian Huang, Xinkun Wang &Ruidong Wang (2017) Strictly convex n-normed spaces and benz theorem, *Journal of Interdisciplinary Mathematics,* 20:5, 1241-1254, DOI: 10.1080/09720502.2017.1334365.

Received October, 2021

Revised December, 2021