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Shape preserving approximation for unbounded multivariate functions

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Abstract

The purpose of this research is to present a new method for creating polynomials and spline approximation to unbounded multivariate functions in $\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$ using ordinary modulus of smoothness, Sendov-Popov modulus and Ditzian-Totik modulus.

Subject Classification: *30E15; 65D12.*

Keywords: *Unbounded functions, Shape preserving approximation, Modulus of smoothness.*

1. Introduction

Let $\mathbb{L}_{\mathcal{P},\mathcal{B}}[-1,1]^{\mathcal{I}}$ be a set of measurable multivariate functions on $[-1,1]^{\mathcal{I}}$, $1 \le \mathcal{P} < \infty$, where

$$\|\lambda\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \left(\int_{-1}^{1}\cdots\int_{-1}^{1}|\lambda(t_1,\ldots,t_{\mathcal{I}})|^{\mathcal{P}}\mathrm{d}t_1\ldots\mathrm{d}t_{\mathcal{I}}\right)^{1/\mathcal{P}} < \infty. \quad (1)$$

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$\mathcal{W}(\mathcal{B},(t_1,\ldots,t_{\mathcal{I}}))$ be a set of all weighted multivariate functions on $[-1,1]^{\mathcal{I}}$, where $\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right| \le \mathcal{N}, \mathcal{N} \in \mathcal{R}^+$ and $\mathcal{B}:[-1,1]^{\mathcal{I}} \to \mathcal{R}^+$ be a multivariate weight function $\mathbb{L}_{\mathcal{P},\mathcal{B}}[-1,1]^{\mathcal{I}}$, $1 \le \mathcal{P} < \infty$ be the space of all unbounded multivariate functions which satisfies the following rule

$$\|\lambda\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}} \mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{1/\mathcal{P}} < \infty . \tag{2}$$

In this research, we need to define our modulus of smoothness as follows

$$\omega_n(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \sup_{\substack{0\le\hbar_\ell\le\delta_\ell \\ \ell=1,\ldots,\mathcal{I}}} \|\Delta_{\hbar}^{n}(\lambda,(t_1,\ldots,t_{\mathcal{I}}))\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{3}$$

is the multi *nth* modulus smoothness of $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, where

$$\Delta_{\hbar}^{n}(\lambda,(t_1,\ldots,t_{\mathcal{I}})) = \begin{cases} \sum_{i=0}^{n}\left(\binom{n}{i}(-1)^{n-i}\right)^{\mathcal{I}} \lambda(t_1+i\hbar_1,\ldots,t_{\mathcal{I}}+i\hbar_{\mathcal{I}}) & \text{if } t, t+i\hbar \in [-1,1]^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

is the multi *nth* difference of λ, such that $t=(t_1,\ldots,t_{\mathcal{I}})$ and $t+i\hbar = (t_1+i\hbar_1,\ldots, t_{\mathcal{I}}+i\hbar_{\mathcal{I}})$. And for $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, we denoted the following sendov-popov modulus of multi *nth* order

$$\tau_n(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \|\omega_n(\lambda,(t_1,\ldots,t_{\mathcal{I}}))\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{5}$$

Also, we defined the following multi*nth* order Ditzian-Totik modulus in the $\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$

$$\omega_n^{\varphi}(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \sup_{\substack{0\le\hbar_\ell\le\delta_\ell \\ \ell=1,\ldots,\mathcal{I}}} \|\Delta_{\hbar\varphi}^{n}(\lambda,(t_1,\ldots,t_{\mathcal{I}}))\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}, \tag{6}$$

where $\varphi(t_\ell) = \sqrt{1-t_\ell^2}$, $t_\ell \in [-1,1]$, $\ell = 1,\ldots,\mathcal{I}$. Let $\mathcal{D}_r = \{(\mathfrak{a}_{11},\ldots,\mathfrak{a}_{1\mathcal{I}}), \ldots,(\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}) : \mathfrak{a}_{0\ell} = -1 < \mathfrak{a}_{1\ell} < \ldots < \mathfrak{a}_{r\ell} < \mathfrak{a}_{r\ell+1} = 1\}$, $r \ge 0$ and $\Delta^0(\mathcal{D}_r)$ be the collection of all multivariate functions λ, where $(-1)^{r-j}\lambda\ (t_1,\ldots,t_{\mathcal{I}}) \ge 0$ $\forall t \in [\mathfrak{a}_{j1},\mathfrak{a}_{j1+1}] \times\ldots\times [\mathfrak{a}_{j\mathcal{I}},\mathfrak{a}_{j\mathcal{I}+1}]$, $0 \le j < r$.

In other words, $\forall\lambda \in \Delta^0(\mathcal{D}_r)$, $0 \le r < \infty$ the signal will change at the points in $\mathcal{D}_r$ and with a positive approach to 1. If r = 0, this leads to $\Delta^0 = \Delta^0(\mathcal{D}_r)$ be the collection of all positive multivariate functions on $[-1,1]^{\mathcal{I}}$. A multivariate function $\aleph$ is called copositive with λ if $\lambda(t_1,\ldots,t_{\mathcal{I}})\,\aleph(t_1,\ldots,t_{\mathcal{I}}) \geq 0,\ \forall t=(t_1,\ldots,t_{\mathcal{I}}) \in [-1,1]^{\mathcal{I}}$.

Regarding the approximation, we are concerned to approximate a multivariate functions using polynomials $\in \mathcal{P}_m$ of degree not exceeding m and spline with less than m knots which are copositive with multivariate function λ, if $r = 0$, in this case the approximation is also said to be positive. $\forall \lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, suppose that

$$E_m(\lambda)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \inf\{\|\lambda - \mathfrak{q}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} : \mathfrak{q}_m \in \mathcal{P}_m\} \tag{7}$$

be the degree of best unconstraint approximation of unbounded multivariate function λ using multi polynomial $\mathfrak{q}_m \in \mathcal{P}_m$, where $\mathfrak{q}_m$ has the following form

$$\mathfrak{q}_m(t_1,\ldots,t_{\mathcal{I}}) = (c_{01}+c_{02}+\ldots+c_{0\mathcal{I}})+c_1(t_1+\ldots+t_{\mathcal{I}})+\ldots$$
$$+c_m(t_1^m+\ldots+t_{\mathcal{I}}^m).$$

And we denote by $E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$

$$= \inf\{\|\lambda - \mathfrak{q}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} : \mathfrak{q}_m \in \mathcal{P}_m \cap \Delta^0(\mathcal{D}_r)\} \tag{8}$$

is the degree of copositive approximation of multivariate function $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, specifically $E_m^{(0)}(\lambda)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} =$

$$E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} = \inf\{\|\lambda - \mathfrak{q}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} : \mathfrak{q}_m \in \mathcal{P}_m \cap \Delta^0\} \tag{9}$$

be the degree of the best copositive approximation using multipolynomial $\mathfrak{q}_m \in \mathcal{P}_m$. Many researchers have studied this topic such as [1,2,3,5,6].

2. Notation and The Main Theorems

Suppose that $-1=\mathfrak{a}_{01}<\mathfrak{a}_{11}<\mathfrak{a}_{21}<\ldots<\mathfrak{a}_{n1-1}<\mathfrak{a}_{n1}=1,\ldots,-1=\mathfrak{a}_{0\mathcal{I}}<\mathfrak{a}_{1\mathcal{I}}<\mathfrak{a}_{2\mathcal{I}}<\ldots<\mathfrak{a}_{n\mathcal{I}-1}<\mathfrak{a}_{n\mathcal{I}}=1$, is the divider of $[-1,1]^{\mathcal{I}}$ and $n>0$ be an integer. Also, we define a support knots for that by $\mathfrak{a}_{1\ell}=-1+i\Delta\mathfrak{a}_{0\ell}$, $i=-k+1,\ldots,-1$, and $\mathfrak{a}_{1\ell}=1-(i-n)\Delta\mathfrak{a}_{n\ell-1}$, $i=n+1,\ n+2,\ldots,\ n+k-1$. Let $J=[\mathfrak{a}_{i1},\mathfrak{a}_{i1+1}]\times\ldots\times[\mathfrak{a}_{i\mathcal{I}},\mathfrak{a}_{i\mathcal{I}+1}]$, $I=[\mathfrak{a}_{i1-k+1},\mathfrak{a}_{i1+k}]\times\ldots\times[\mathfrak{a}_{i\mathcal{I}-k+1},\mathfrak{a}_{i\mathcal{I}+k}]$.

Let's denote $\mathcal{S}_{n\ell}=\{\mathfrak{a}_{i\ell}\}_{i=-k+1}^{n+k-1}$, $\Delta\mathcal{S}_{n\ell}=\max\{\Delta\mathfrak{a}_{i\ell}\}$, $\ell=1,\ldots,\mathcal{I}$. Then $\forall i=\ -k+1,\ldots,n-1$ we have $\mathfrak{b}_{i\ell}=\frac{\mathfrak{a}_{i\ell+k}-\mathfrak{a}_{i\ell}}{k}$, $\mathfrak{a}_{i\ell}^*=\frac{(\mathfrak{a}_{i\ell+1},\ldots,\mathfrak{a}_{i\ell+k-1})}{k-1}$, $\mathfrak{b}_{i\ell}^*=2\min\frac{(\mathfrak{a}_{i\ell}^*-\mathfrak{a}_{\ell},\mathfrak{a}_{i\ell+k}-\mathfrak{a}_{i\ell}^*)}{k}$, and $\tilde{J}=\left[\mathfrak{a}_{i1}^*-\frac{\mathfrak{b}_{i1}^*}{2},\mathfrak{a}_{i1}^*+\frac{\mathfrak{b}_{i1}^*}{2}\right]\times\ldots\times$

$\left[\mathfrak{a}^*_{i\mathcal{I}} - \frac{\mathfrak{b}^*_{i\mathcal{I}}}{2}, \mathfrak{a}^*_{i\mathcal{I}} + \frac{\mathfrak{b}^*_{i\mathcal{I}}}{2}\right]$. Also, we will define the operator **S** for all $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}$ $([-1,1]^{\mathcal{I}})$ by $\mathcal{S}(\lambda) = \sum_{i=-k+1}^{n-1}(\Lambda_{i1}\ \Omega_{i1} + \Lambda_{i2}\Omega_{i2} + \ldots + \Lambda_{i\mathcal{I}}\Omega_{i\mathcal{I}})$, where $\Omega_i(t) = \Omega_{i,k}(t) = \Omega(t, (\mathfrak{a}_{i1}, \ldots, \mathfrak{a}_{i\mathcal{I}}), \ldots, (\mathfrak{a}_{i1+k}, \ldots, \mathfrak{a}_{i\mathcal{I}+k}))$ on $((\mathfrak{a}_{i1}, \ldots, \mathfrak{a}_{i\mathcal{I}}), \ldots, \mathfrak{a}_{i1+1}, \ldots, \mathfrak{a}_{i\mathcal{I}+1})$ standardized so that $\sum_i^{i+k}\Omega_{i\ell} = 1$.

Now, through the following theorems we present the results of this research:

Theorem 2.1 : *Let* $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, $1 \le \mathcal{P} < \infty$. *Then there exists a spline* $\psi = \mathcal{S}(\lambda)$ *of degree m on the knots sequence* $\mathcal{S}_n$ *satisfies*

$$E_m^0(\lambda, \mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \le \Lambda_{k,\mathcal{I}}\omega_2(\lambda, \delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{10}$$

such that $\delta_\ell = \min_{0\le i\le r}|\mathfrak{a}_{i\ell+1} - \mathfrak{a}_{i\ell}|$ *and* $\Lambda_{k,\mathcal{I}}$ *is a constant depends on k and* $\mathcal{I}$.

Proof : Using Holder's inequality, we obtain $|\Lambda_{j1}\ldots\Lambda_{j\mathcal{I}}| \le \left(\sum_{j\in[-1,1]^{\mathcal{I}}}(\Lambda_{i1}\ \Omega_{i1} + \Lambda_{i2}\Omega_{i2} + \ldots + \Lambda_{i\mathcal{I}}\Omega_{i\mathcal{I}})\right) \le \left(\sum_{j\in[-1,1]^{\mathcal{I}}}|\ \Lambda_{j1}\ldots\Lambda_{j\mathcal{I}}|^{\mathcal{P}}\right)^{\frac{1}{\mathcal{P}}}.\ \left(\sum_{j\in[-1,1]^{\mathcal{I}}}|\ \Omega_{j1}\ldots\Omega_{j\mathcal{I}}|^{q}\right)^{\frac{1}{q}}$. Since, $\sum_{j\in[-1,1]^{\mathcal{I}}}\Omega_{j\ell} = 1$ and $\int_{-\infty}^{+\infty}\Omega_{j\ell}\mathrm{dt}_\ell = \mathfrak{b}_{j\ell}$, we conclude that

$$|\Lambda_{j1}\ldots\Lambda_{j\mathcal{I}}| \le \Lambda\|\lambda\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{11}$$

By [4], we get

$$\begin{aligned}
\|\mathcal{S}(\lambda)\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} &= \left(\int_{-1}^{1}\ldots\int_{-1}^{1}\left|\frac{\mathcal{S}(\lambda,(t_1,\ldots,t_{\mathcal{I}}))}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}} \\
&= \left(\int_{-1}^{1}\ldots\int_{-1}^{1}\left|\frac{\sum_j^i(\Lambda_{j1}\Omega_{j1}+\Lambda_{j2}\Omega_{j2}+\ldots+\Lambda_{j\mathcal{I}}\Omega_{j\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}} \\
&\le \Lambda\left(\int_{-1}^{1}\ldots\int_{-1}^{1}\left|\frac{\sum_j^i(\Lambda_{j1}\mathfrak{b}_{j1}+\Lambda_{j2}\mathfrak{b}_{j2}+\ldots+\Lambda_{j\mathcal{I}}\mathfrak{b}_{j\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}} \\
&\le \Lambda_1\|\lambda\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}
\end{aligned} \tag{12}$$

Let g_i^* is the best approximation to multivariate function on $[-1,1]^{\mathcal{I}}$, this leads to that g_i^* is the best approximation on all $\hat{I} \subset [-1,1]^{\mathcal{I}}$ and using [7], we get

$$\|\lambda - g_i^*\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \le \Lambda\omega_2(\lambda, \delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{13}$$

By virtue of (11), (12) and (13) we get

$$E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$$

$$\leq \|\lambda-\psi\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \leq \Lambda\|\lambda-\mathcal{S}(\lambda)\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$$

$$\leq \Lambda_1\left\{\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\lambda(t_1,\dots,t_{\mathcal{I}})-g_i^*(t_1,\dots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\dots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\int_{-1}^{1}\mathrm{dt}_1\dots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\right.$$

$$\left.+\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{g_i^*(t_1,\dots,t_{\mathcal{I}})-\mathcal{S}(\lambda,(t_1,\dots,t_{\mathcal{I}}))}{\mathcal{B}(t_1,\dots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\dots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\right\}$$

$$\leq \Lambda_1\left\{\Lambda_2\omega_2(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}+\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\mathcal{S}(g_i^*,(t_1,\dots,t_{\mathcal{I}}))-\mathcal{S}(\lambda,(t_1,\dots,t_{\mathcal{I}}))}{\mathcal{B}(t_1,\dots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\dots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\right\}$$

$$\leq \Lambda_3\omega_2(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}+\Lambda_3\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\lambda(t_1,\dots,t_{\mathcal{I}})-g_i^*(t_1,\dots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\dots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\dots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}$$

$$\leq \Lambda_4\omega_2(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}+\Lambda_5\omega_2(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\leq \Lambda_6\omega_2(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}.$$

Therefore (10) is verified.

Theorem 2.2 : *Let* $\mathcal{D}_r=\{(\mathfrak{a}_{11},\dots,\mathfrak{a}_{1\mathcal{I}}),(\mathfrak{a}_{21},\dots,\mathfrak{a}_{2\mathcal{I}}),\dots,(\mathfrak{a}_{r1},\dots,\mathfrak{a}_{r\mathcal{I}})\}$, if $\lambda\in\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$, $1\leq\mathcal{P}<\infty$, *then*

$$E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\leq\Lambda\omega_n^{\varphi}(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})},\tag{14}$$

where $\delta_\ell=\min_{0\leq i\leq r}|\mathfrak{a}_{i\ell+1}-\mathfrak{a}_{i\ell}|$, *and* Λ *is a constant*.

***Proof* :** Suppose that for all multivariate functions $\lambda\in\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})\bigcap\Delta^0(\mathcal{D}_r)$, then (14) is true, using [5], there is spline $\eta\in\Delta^0(\mathcal{D}_r)$ with knots $\{\mathfrak{a}_{j\ell}\}_{j=0}^m$ satisfying (13). Let $\mathcal{G}(t_1,\dots,t_{\mathcal{I}})=\eta(t_1,\dots,t_{\mathcal{I}})\,\alpha(t_1-\mathfrak{a}_{r1},\dots,t_{\mathcal{I}}-\mathfrak{a}_{r\mathcal{I}})$, therefore $\mathcal{G}(t_1,\dots,t_{\mathcal{I}})\in\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})\bigcap\Delta^0(\mathcal{D}_r)$ and using the assumption, there is $\psi_m\in\mathcal{P}_m\bigcap\Delta^0(\mathcal{D}_{r-1})$, where

$$\|\mathcal{G}-\psi_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\leq\Lambda\omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})},\zeta=\left(\frac{1}{n},\dots,\frac{1}{n}\right)\tag{15}$$

We have $\mathcal{H}_m(t_1,\ldots,t_\mathcal{I})=\psi_m(t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),\ (t_1,\ldots,t_\mathcal{I}))$, such that $\mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),\ (t_1,\ldots,t_\mathcal{I}))$ a multipolynomial that is copositive with $\eta\ (t_1-\mathfrak{a}_{r1},\ldots,\ t_\mathcal{I}-\mathfrak{a}_{r\mathcal{I}})$, Obviously, $\mathcal{H}_m(t_1,\ldots,t_\mathcal{I})\in\mathcal{P}_m\bigcap\Delta^0(\mathcal{D}_r)$.

We want to estimate $\|\mathcal{G}-\mathcal{H}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}$ and $\|\lambda-\mathcal{H}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}$. Using (15), we get

$$\|\mathcal{G}-\mathcal{H}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}$$

$$=\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\mathcal{G}(t_1,\ldots,t_\mathcal{I})-\mathcal{H}_m(t_1,\ldots,t_\mathcal{I})}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

$$=\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\eta\ (t_1,\ldots,t_\mathcal{I})\alpha\ (t_1-\mathfrak{a}_{r1},\ldots,t_\mathcal{I}-\mathfrak{a}_{r\mathcal{I}})-\psi_m(t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

$$\leq\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\eta\ (t_1,\ldots,t_\mathcal{I})\alpha\ (t_1-\mathfrak{a}_{r1},\ldots,t_\mathcal{I}-\mathfrak{a}_{r\mathcal{I}})-\eta\ (t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

$$+\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\eta\ (t_1,\ldots,t_\mathcal{I})\mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))-\psi_m(t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

$$\leq\mathcal{Z}$$

$$+\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\eta\ (t_1,\ldots,t_\mathcal{I})\mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))-\psi_m(t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

Where

$$\mathcal{Z}=\left(\int_{-1}^{1}\cdots\int_{-1}^{1}\left|\frac{\eta\ (t_1,\ldots,t_\mathcal{I})\alpha\ (t_1-\mathfrak{a}_{r1},\ldots,t_\mathcal{I}-\mathfrak{a}_{r\mathcal{I}})-\eta\ (t_1,\ldots,t_\mathcal{I})\ \mathfrak{q}_m((\mathfrak{a}_{r1},\ldots,\mathfrak{a}_{r\mathcal{I}}),(t_1,\ldots,t_\mathcal{I}))}{\mathcal{B}(t_1,\ldots,t_\mathcal{I})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_\mathcal{I}\right)^{\frac{1}{\mathcal{P}}}$$

Using [6] and the properties of Ditzian-Totik modulus, we get $\mathcal{Z}\leq\Lambda\omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}$. Therefore,

$$\|\mathcal{G}-\mathcal{H}_m\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}\leq\Lambda_1\omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})}+\Lambda_2\omega_n^{\varphi}(\eta,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^\mathcal{I})},$$

since $\omega_n^{\varphi}(\eta,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \leq \omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$, then $\| \mathcal{G}-\mathcal{H}_m \|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$ $\leq \Lambda_3 \omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$. And

$$\begin{aligned}\| \lambda-\mathcal{H}_m \|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} &\leq \left(\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\mathcal{G}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})} \right|^{\mathcal{P}} \mathrm{dt}_1 \ldots \mathrm{dt}_{\mathcal{I}} \right)^{\frac{1}{\mathcal{P}}} \\ &+ \left(\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\mathcal{G}(t_1,\ldots,t_{\mathcal{I}})-\mathcal{H}_m(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})} \right|^{\mathcal{P}} \mathrm{dt}_1 \ldots \mathrm{dt}_{\mathcal{I}} \right)^{\frac{1}{\mathcal{P}}} \\ &\leq \Lambda_4 \omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} + \Lambda_5 \omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})},\end{aligned}$$

since $\omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \leq \omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$, thus $\| \lambda-\mathcal{H}_m \|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \leq \Lambda_6 \omega_m^{\varphi}$ $(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$. Therefore

$$\begin{aligned}E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} &\leq \| \lambda-\psi_m \|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \\ &\leq \left(\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\mathcal{H}_m(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})} \right|^{\mathcal{P}} \mathrm{dt}_1 \ldots \mathrm{dt}_{\mathcal{I}} \right)^{\frac{1}{\mathcal{P}}} \\ &+ \left(\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\mathcal{H}_m(t_1,\ldots,t_{\mathcal{I}})-\mathcal{G}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})} \right|^{\mathcal{P}} \mathrm{dt}_1 \ldots \mathrm{dt}_{\mathcal{I}} \right)^{\frac{1}{\mathcal{P}}} \\ &+ \left(\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\mathcal{G}(t_1,\ldots,t_{\mathcal{I}})-\psi_m(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})} \right|^{\mathcal{P}} \mathrm{dt}_1 \ldots \mathrm{dt}_{\mathcal{I}} \right)^{\frac{1}{\mathcal{P}}}.\end{aligned}$$

$$\leq \Lambda_7 \omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} + \Lambda_8 \omega_n^{\varphi}(\mathcal{G},\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} + \Lambda_9 \omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$$

$\leq \Lambda_{10} \omega_n^{\varphi}(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$. Thus (14) is proved.

Theorem 2.3 : *Let* $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}}) \cap \Delta^0(\mathcal{D}_r)$, $1 \leq \mathcal{P} < \infty$, *change its positivity finitely many times, say* n, *at* $-1 = \mathfrak{a}_{01} < \mathfrak{a}_{11} < \ldots < \mathfrak{a}_{n1} < \mathfrak{a}_{n1+1} = 1, \ldots, -1 = \mathfrak{a}_{0\mathcal{I}} < \mathfrak{a}_{1\mathcal{I}} < \ldots < \mathfrak{a}_{n\mathcal{I}} < \mathfrak{a}_{n\mathcal{I}+1} = 1$ *and* $\delta_{\ell} = \min_{0 \leq i \leq r} |\mathfrak{a}_{i\ell+1} - \mathfrak{a}_{i\ell}|$, $\delta_{\ell} > \frac{1}{n}$, *then there is a quadratic spline* ψ_m *with nth knots which copositive with* λ *and satisfies*

$$E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \leq \Lambda \tau_3(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} \tag{16}$$

Proof : Let $\mathcal{F}_{i\ell}=\frac{i}{n}$, $\hat{I}_i=[\mathcal{F}_{i1},\mathcal{F}_{i1+1}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}},\mathcal{F}_{i\mathcal{I}+1}]$, if $\mathcal{F}_{i\ell}<\mathfrak{a}_{i\ell}<\mathcal{F}_{i\ell+1}$, there is a sign change in $\mathfrak{a}_{i\ell}$ from λ, $1\le i\le n$. In each of suitability, there is one $\mathfrak{a}_{i\ell}$ for all $\delta_\ell>\frac{1}{n}$, correspondingly , let $\hat{I}_{0\ell}=-1$ and $\hat{I}_{n\ell+1}=2m$, such that $\hat{I}_{i\ell}<\hat{I}_{i\ell}+2<\hat{I}_{i\ell+1}$, $0\le i\le n$, which is the middle of $I_{\hat{I}_{i\ell}}$ and $I_{\hat{I}_{i\ell+1}}$. for all $0\le i\le n$ there is at least one period $\hat{I}_i$ which is uncontaminated , and λ does not change its sign at the middle of $I_{\hat{I}_{i\ell}}$ and $I_{\hat{I}_{i\ell+1}}$. If $\hat{I}_{i\ell+1}>\hat{I}_{i\ell}+2$, there are at least 2 uncontaminated periods in the middle of $I_{\hat{I}_{i\ell}}$ and $I_{\hat{I}_{i\ell+1}}$. We have $\mathcal{A}_i$ and $\mathcal{V}_i$ two multi-polynomials such that $\mathcal{V}_i\le\lambda\ (\mathcal{F}_1,\ldots,\mathcal{F}_{\mathcal{I}})\le\mathcal{A}_i$, $\mathcal{F}\in[\mathcal{F}_{i1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}},\mathcal{F}_{i\mathcal{I}+2}]$ and

$$\|\mathcal{A}_i-\mathcal{V}_i\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}=\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\mathcal{A}_i(t_1,\ldots,t_{\mathcal{I}})-\mathcal{V}_i(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}$$

$\le\Lambda\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$, $\zeta=\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$, $t\in[\mathcal{F}_{i1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}},\mathcal{F}_{i\mathcal{I}+2}]$ and $\mathcal{A}_i$ is a copositive with λ greater than or equal to 0 and $\mathcal{V}_i$ is a copositive with λ less than or equal to 0 and $\|\mathcal{A}_i-\lambda\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\le\|\mathcal{A}_i-\mathcal{V}_i\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$, $\|\lambda-\mathcal{V}_i\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\le\|\mathcal{A}_i-\mathcal{V}_i\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$.

Using interpolation on $[\mathcal{F}_{i1-1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}-1},\mathcal{F}_{i\mathcal{I}+2}]$, we make a multi-polynomial locally and use multi-polynomial $\Omega_{\hat{I}_{i-1}}$, its copositive with λ on $[\mathcal{F}_{i1-1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}-1},\mathcal{F}_{i\mathcal{I}+2}]$, we take the interpolation of λ on $[\mathcal{F}_{i1-1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}-1},\mathcal{F}_{i\mathcal{I}+2}]$. As for the rate of approximation , we will take two multipolynomials $\mathcal{A}_{\hat{I}_{i-1}}$ and $\mathcal{V}_{\hat{I}_{i-1}}$, such that $\mathcal{V}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})$ $\le\lambda(t_1,\ldots,t_{\mathcal{I}})\le\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})$, $t\in[\mathcal{F}_{i1-1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}-1},\mathcal{F}_{i\mathcal{I}+2}]$.

Because

$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})-\Omega_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}$$
$$\le\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})-\mathcal{V}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}$$
$$\le\ddot{\mathrm{E}}\,\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})},t\in[\mathcal{F}_{i1},\mathcal{F}_{i1+2}]\times\ldots\times[\mathcal{F}_{i\mathcal{I}},\mathcal{F}_{i\mathcal{I}+2}]$$

Hence,

$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\Omega_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\le$$

$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}+$$
$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})-\Omega_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\leq$$
$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}+$$
$$\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\mathcal{A}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})-\mathcal{V}_{\hat{I}_{i-1}}(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\leq$$
$$\Lambda_1\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}+\Lambda_2\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\leq$$
$$\Lambda_3\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}.$$

Thus, we obtain local multipolynomials that have an estimated degree of 3 and are copositive with λ. At the same time, we superimpose them to obtain an approximation of *nth* spline ψ with a similar degree of estimation. If $J=[\mathfrak{a}_{i1},\mathfrak{a}_{i1+1}]\times\ldots\times[\mathfrak{a}_{i\mathcal{I}},\mathfrak{a}_{i\mathcal{I}+1}]$ is uncontaminated period, then $X=[-1,1]^{\mathcal{I}}$ must also be uncontaminated. In addition, ψ_i is also copositive with λ and

$$\begin{aligned}\|\lambda-\psi_i\|_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})} &\leq \left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\lambda(t_1,\ldots,t_{\mathcal{I}})-\Omega_i(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\\
&+\left(\int_{\mathcal{F}_{i1}}^{\mathcal{F}_{i1+2}}\ldots\int_{\mathcal{F}_{i\mathcal{I}}}^{\mathcal{F}_{i\mathcal{I}+2}}\left|\frac{\Omega_i(t_1,\ldots,t_{\mathcal{I}})-\psi_i(t_1,\ldots,t_{\mathcal{I}})}{\mathcal{B}(t_1,\ldots,t_{\mathcal{I}})}\right|^{\mathcal{P}}\mathrm{dt}_1\ldots\mathrm{dt}_{\mathcal{I}}\right)^{\frac{1}{\mathcal{P}}}\\
&\leq \Lambda_1\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}+\Lambda_2\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\\
&\leq \Lambda_3\tau_3(\lambda,\zeta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}.\end{aligned}$$

Hence, $E_m^{(0)}(\lambda,\mathcal{D}_r)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}\leq\Lambda\tau_3(\lambda,\delta)_{\mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})}$.

3. Conclusion

In this paper, we deal with copositive spline approximation of unbounded multivariate functions using a second – order smoothness modulus for $1\leq\mathcal{P}\leq\infty$. We also showed that it is easy to get the Ditzian-Totik modulus, therefore belongs to the degree of copositive polynomials approximation of unbounded multivariate functions. Finally, we construct the inequality in terms Sendov-Popov modulus of copositive spline approximation to $\lambda \in \mathbb{L}_{\mathcal{P},\mathcal{B}}([-1,1]^{\mathcal{I}})$.

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