# In Predictable Outcome of Some Complex Function on l_2 Space

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## Abstract

The Main advantage of this work is to concentrate on the outcome of the function $\mu_\gamma : l^p \to l^p$ as a function of $\zeta(s)$ , ( $s = \sigma + it$ , $\sigma , t \in \Re$ ) . This type of work has been studied deeply by [HIL , 12] . Here , we see that the outcomes of the function $\mu_\gamma$ are depending on the of the function $\zeta(s)$ . More deeply , $\mu_\gamma$ ' s value will appear as a matrix of zero triangle values depending 0n the positive of $\sigma$ on the real line .

**Key words:** Dirichlet function, Functional analysis.

## الخلاصة

المنفعة الاساسية من هذا العمل هي التركيز على مخرجات الدالة $\mu_\gamma : l^p \to l^q$ كدالة بدلالة $\zeta(s)$ (حيث $t, \sigma$ , $s = \sigma + it$ أعداد حقيقية) هذا الموضوع درس عميقا من قبل العالم هلبيردنك. [HIL, 12 ] هنا نرى بأن مخرجات الدالة $\mu_\gamma$ تكون معتمدة على $\zeta(s)$ أكثر تعمقا قيم $\mu_\gamma$ سوف تظهر كمصفوفة ذات مثلثية صفرية القيم معتمدة على موقع $\sigma$ على خط الاعداد .

**الكلمات المفتاحية:** التحليل الدالي، دالة دريشليه.

## 1. Introduction

Firstly, we start in this introduction by viewing some definitions of the arithmetical functions and norm function [AI, 85]. Secondly, we assert some of basic (known) theorems without proof related to the upper and lower bounds of $\zeta(s)$.
An arithmetical function is a function $f: N \to C$. Denote by $A$ the set of all arithmetical functions. For $f, g \in A$ $\lambda \in C$ we have $\lambda f$, $f + g$ and $fg$ are also in $A$. More importantly, for any arithmetical functions we define $f * g$ by

$$(f*g)(n) = \sum_{d|n} f(d) g\left(\frac{n}{d}\right)$$

is also in $A$. The sum here is over all divisors $d$ of $n$. Dirichlet convolution is commutative. This follows from the sum

$$(f*g)(n) = \sum_{d|n} f(d) g(\tfrac{n}{d}) = \sum_{c,d>1} f(c) g(d),$$

where the sum is over all possible positive integers $c, d$ such that $cd = n$. It is also associative since

$$f(*g*h)(n) = \sum_{bcd=n} f(b) g(c) h(d).$$

In fact $(A, +, *)$ is an algebra where $*$ is distributive with respect to $+$ and $\lambda(f * g) = (\lambda f) *g = f * (\lambda g)$ for every $\lambda \in C$. Now, we move our attention to define some arithmetical functions. We start with the divisor function $d(n)$.

**1-1 Definition :-**

For $n \in N$, we define the function $d(n)$ to be the number of divisors of $n$. We write this as $d(n) = \sum_{d|n} 1$, where the sum ranges over all the divisors $d$ of n.

**1-2 Definition :-** [Al,1985]

For $\Re e\ s > 1$, we defin

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Here the above summation converges absolutely and locally uniformly for $\Re e\ s > 1$ **(see [AI, 85]).** Moreover, $\zeta(s)$ has an analytic continuation to the whole complex plane except for a simple pole at 1 with residue 1 and is of finite order which means that $|\zeta(s)| \leq c\, t^{\delta}$, for some $c, \delta > 0$ where $\delta$ depends on the real part of $s$ (see [LHIL, 06]).

The function $\zeta(s)$ is a function of complex variable that has been studied by B. Riemann (1826-1866). There is an important link between $\zeta(s)$ and the prime numbers. This is clear from the following:

$$\zeta(s) = \prod_p (1 - \frac{1}{p^s})^{-1}.$$

Here, product holds for the real part of $s$ must greater than 1.

The zeros of $\zeta(s)$ for $0 \leq \Re e\ s \leq 1$ is an important subject and has a significance conjectures (see [AI, 85]). Riemann showed that the frequency of prime numbers is very closely related to the behavior of the zeros of $\zeta(s)$. He conjectured that all non-trivial zeros of $\zeta(s)$ lies on a line have the real part $\frac{1}{2}$.

**1-3 Theorem (Basic Properties):** For $s = \sigma + it$, we have

(a) $\zeta(s)$ is analytic in $\sigma>1$.

(b) $\zeta(s)$ has an analytic continuation to the half plane $\sigma > 0$ except at the simple pole $s = 1$ with residue 1. We mean by Analytic continuation that for $\sigma > 0$, we have

$$\zeta(s) = \frac{s}{s-1} - s\int_1^{\infty} \{x\} x^{-s-1}\, dx.$$

(c) $\zeta(s)$ has an Euler product representation for $\sigma > 1$

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1}$$

(d) $\zeta(s)$ has no zeros for $\sigma > 1$.

For the proof of the above axioms see for example [Al,1976].

**Remark:** For more details and other (basic) information about $\zeta(s)$ see **[TA, 1976], [MN,2005] and [PB, 2004].**

As a part of this investigation, we need some functions from the functional analysis (which we use later on to prove the main theorem). We start with the definition of the space

$$l^2 = \left\{(a_n)_{n\in N}: \sum_{n=1}^{\infty} |a_n|^2 < \infty\right\},$$

and we call it $l^2$ space. Let the function $\mu_\gamma: l^2 \to l^2$ be a linear mapping defined as follows $\mu_\gamma(a_n) = (b_n)$ where $b_n = \sum_{d|n} \left(\frac{n}{d}\right)^{-\gamma} a_\gamma$. the important question here is : When is the function $\mu_\gamma: l^2 \to l^2$ bounded? From the work which has been done in ***[HIL, 09],*** we understand that $\mu_\gamma$ is bounded if and only if $\gamma > 1$, in which case $\|\mu_\gamma\| = \zeta(\gamma)$. To see this, we have

$$|b_n|^2 = \left|\sum_{d|n} \left(\frac{n}{d}\right)^{-\gamma}\right|^2 = \frac{1}{n^{2\gamma}}\left|\sum_{d|n} d^\gamma a_\gamma\right|^2.$$

By Cauchy–Schwarz inequality we have the last term is

$$\leq \frac{1}{n^{2\gamma}}\left(\sum_{d|n} d^\gamma\right)\left(\sum_{d|n} d^\gamma |a_d|^2\right) = \frac{1}{n^\gamma}\left(\sum_{d|n}\frac{1}{d^\gamma}\right)\left(\sum_{d|n} d^\gamma |a_d|^2\right).$$

Therefore, we have

$$|b_n|^2 \leq \frac{\zeta(\gamma)}{n^\gamma}\sum_{d|n} d^\gamma |a_d|^2. \text{ Hence,}$$

$$\sum_{n=1}^{\infty}|b_n|^2 \leq \zeta(\gamma)\sum_{n=1}^{\infty}\frac{1}{n^\gamma}\sum_{d|n} d^\gamma |a_d|^2 = (\zeta(\gamma))^2\sum_{d=1}^{\infty}|a_d|^2.$$

This means that

$$\|b\|^2 \leq \left(\zeta(\gamma)\right)^2\|a\|^2.$$

Which tells us that $\|\mu_\gamma(a)\| \leq \zeta(\gamma)\|a\|$, and from the knowledge of $\zeta(s)$ we see that the function $\mu_\gamma$ is bounded for $\gamma > 1$ and $\|\mu_\gamma\| \leq \zeta(\gamma)$.

$\mu_\gamma: l^2 \to l^2$ is linear mapping on $l^2$

We see here that $b_n = \sum_{m=0}^{\infty} e_{nm} a_m$ , where $e_{nm}$ are coefficients . that is

$$E = (e_{nm})_{m,n\geq 1} = \begin{pmatrix} e_{11} & e_{12} & \cdots \\ e_{21} & e_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$$

Moreover , it is straight forward , that $\mu_\gamma \sim E == \begin{pmatrix} e_{11} & e_{12} & \cdots \\ e_{21} & e_{22} & \cdots \\ \vdots & \vdots & \end{pmatrix}$

In this article we ask an important question: When is $\mu_\gamma : l^p \to l^q$ bounded?

We answer the above question after the following theorem. We remark here that there is no loss if we mention the proof of the following theorem in order to show that in which part (of the Euclidian plane) that $\mu_\gamma$ should be bounded. That is, the strict lower bound of the Riemann-zeta function will enable us to determine the area of boundedness of $\mu_\gamma$ .

**2-1 Theorem:**

Let $s$ be a complex number. Then for $\frac{1}{2}+\varepsilon \leq \Re e\, s \leq 1-f(X)$, $(\varepsilon > 0)$, we have for the imaginary part of $s$ runs between 1 and $X$ that the maximum of

$$\log\ |\zeta|\quad \geq c\ \frac{exp\left\{\frac{logloglog\ x}{2}\right\}}{logloglog\ x},$$

where $c$ is a small constant and $X$ sufficiently large ***independent*** of $\sigma$. Here $f(X) = \frac{logloglog\ X}{2loglog\ X}$.

**Note that:** More generally, the above theorem is valid for any function $f(X) \to 0$ as $X \to \infty$.

In order to prove the above theorem we need the following **Lemma**.

**2-2 Lemma:**

For $1-c \geq \frac{logloglog\ T}{2loglogT}$ we have

$$logloglog\ P \sum_{p<P} p^{-\rho} \geq k\ exp\left\{\frac{logloglog\ P}{2}\right\}$$

For $P > P_0$.

**Proof:**

Using [HIL,2012], we see for $\pi(x) = \sum_{p \leq x} 1$, that

$$\sum_{p\leq P} p^{-\rho} = \int_2^P u^{-\rho} d\pi(u).$$

So, by the 'Prime Number Theorem' (which means the assertion that the number of primes that $\pi(x)(= \sum_{p<x} 1) \sim \frac{x}{\log x}$)), we observe the last integral is greater than

$$\frac{e^{(1-\rho)\log P}}{c(1-\rho)\log P} - k,$$

For some constants $c, k > 0$. The result follow immediately.

Now we could start prove the theorem.

## 3. Proof (of the theorem)

For any real $\rho$ running between $\frac{1}{2}+\delta$ and the line 1 (any $\delta > 0$), we see (by ***[AIL,2012])*** that for the imaginary part of s runs between 1 and X the maximum of

$$\zeta(s) \geq \underset{\|a\|_2}{Sup} \left(\sum_{k=1}^{X} |b_k|^2\right)^{\frac{1}{2}},$$

where $a$ is in $l^2$ and $b_n = \sum_{d|n} \left(\frac{n}{d}\right)^{-\rho} a_\rho$. We note that if we take $X$ to be as large as possible and define

$$a_n = \begin{Bmatrix} \frac{1}{\left(\sum_{d|X} 1\right)^{\frac{1}{2}}} & if\ n|X \\ 0 & otherwise \end{Bmatrix},$$

Then we see that

$$\|a\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = \frac{\sum_{d|X} 1}{\sum_{d|X} 1} = 1.$$

$$\|b\|^2 = \sum_{n=1}^{\infty} \|b_n\|^2 \geq \sum_{n|X} {b_n}^2 \geq \frac{\sum_{d|X} d^{-\rho}}{\left(\sum_{d|X} 1\right)^{\frac{1}{2}}} = F_\rho(X).$$

Now, for

$$n = \prod_{p=2}^{P} p$$

We take $X$ to be the largest number lying between $n$ and $n \cdot P_0$ where $P_0$ is the next prime after $P$ [WZ, 07]. Therefore, one can calculate $F_\rho(n)$ to observe

$$F_\rho(n) \geq \sum_{p<P} p^{-\rho} - \theta,$$

For some absolute constant $\theta$. By the Prime Number Theorem [PB, 04], we see that $log\, n \sim P \sim \log X$.

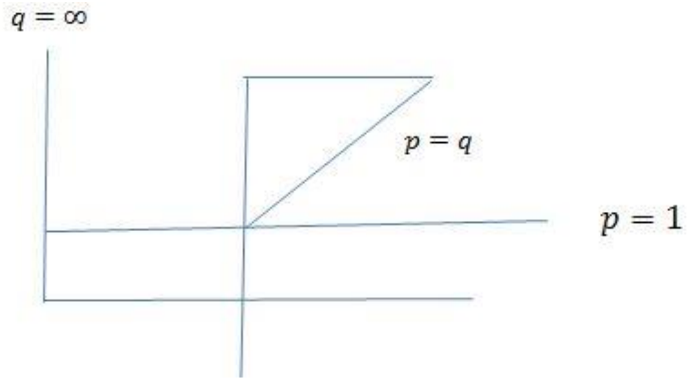
Hence the above lemma finished the proof of the main theorem.

In order to answer the main question mentioned , we see that the associated matrix to $\mu_\gamma$ would be $b_n = \sum_{m=0}^{\infty} e_{nm} a_m = \sum m/_{n} \quad \frac{m^r}{n^r}\, a_m$ .

Where , $e_{nm} = \left[ \begin{array}{lll} (\frac{m}{n})^r & if & m/n \\ o & if & m \not/ n \end{array} \right.$ . That is ,

$$(e_{nm})_{m,n\geq 1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ \frac{1}{2^\gamma} & 1 & 0 & 0 & \dots \\ \frac{1}{3^\gamma} & 0 & 1 & 0 & \dots \\ \frac{1}{4^\gamma} & \frac{1}{2^\gamma} & 0 & 1 & \\ & & & & \ddots \end{pmatrix}.$$

*Thus after some calculation one could set that* $\|\mu_\gamma\| = \sup_{\|a\|=1} \|\mu_\gamma a\| = \zeta(\gamma)$ *for* $\gamma > 1$ *.*
*Therefore the knowledge of* $\zeta(s)$ *tell us that for* $\gamma \leq 1$ *,* $\mu_\gamma$ *is unbounded .*
*hence by Hilberding* $[HIL, 09]$ *we see that if* $p = 1$ *or* $q = \infty$ *or* $p = q$ *,* $\mu_\gamma : l^p \to l^q$ *is bounded with Norm* $\|\mu_\gamma\| \leq \sqrt[\gamma]{\zeta(r\gamma)}$ *for* $\gamma > \frac{1}{r}$ *, where* $\frac{1}{r} = 1 - \frac{1}{p} + \frac{1}{q}$ *see the Figure below*




***For example*** *: for* $p = 2$ *and* $q = 3$ *we see that* $\frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ *( i.e* $r = \frac{6}{5} > 1$)
*this tells us that* $\mu_\gamma : l^2 \to l^3$ *is bounded*
*if* $r > \frac{6}{5}$ *and* $\|\mu_\gamma\| \leq (\zeta(\frac{6}{5}\gamma)^{\frac{5}{6}}$.
*One may ask: Does* $\mu_\gamma$ *bounded for* $r \leq \frac{5}{6}$ *?*

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