New Extrapolation Rule to Increase the Accuracy of Numerical Integrals Results

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Abstract--- This paper discusses a new method to increase the accuracy of numerical integrals results for integrations of continuous functions or continuous functions but their derivatives are obsolete or are themselves obsolete at one end or both boundary of integration by using the error formulas associated with the Newton-Cotes integrative rules, obtained good results in improving integrality values in terms of accuracy and number of periods of fragmentation.

Keywords--- Numerical Integral, Richardson Extrapolation, Aitkendelta- Squared Process.

I. Introduction

Many researchers studied the field of numerical integration and were interested in finding the approximate values of integrals numerically in the form of convergent series to the exact value of single integration and others, quite a few researchers have moved to accelerate the converge of the approximate value to the exact value of integration. In 1910, the English scientist Louis Fry Richardson introduced the method of improving the results of numerical integration, namely the Richardson extrapolation method of solving numerical integration problems and numerical solutions of differential equations where it was based on the division of the range into an equal number of partial periods, a method applied to many of the issues in which the error can be written in the form of a power series dependent on the length of partial period. In 1926, Alexander Aitken created a formula to speed up the successive approximation of the value of an issue called the Aitken's delta - squared process, the result of this formula is to accelerate the approximation of the resulting values in any numerical integrative way to the exact value. It should be noted that the first version of the method is known by the Japanese sports world Takakazu Seki kowa, who lived at the end of the seventeenth century. Brezinski[5].

In1955, the German scientist Romberg Werner introduced a method based on the formation of a triangular arrangement containing numerical approximation values for integration, Rombergrelied on Richardson's method repeatedly on the values of numerical integration by multiplying the number of periods of fragmentation from its predecessors each time. Mohammed[8]. In 2010, researcher Nasser,[9] used the formula of the Aitken interception repeatedly, and the results were better in terms of accuracy and speed of approach. In 2012, Alsharify[3] applying Richardson method with double integrals. In the same year

Al-Karamy[1] used a number of correction limits rather than just the dominant limit of the error formula series when applying the Richardson base, and obtained better results than the first method in terms of accuracy and number of partial periods. Al-Karamy In 2017, Al-Karamy[2] also presented an experimental formula in which Al-Karamy improved the results of the numerical integrations based on the integrative Romberg formula and obtained good results in accuracy and speed of approach.

Aitken's Delta – Squared Process

Assume that $\{a_n\}$ be a sequence of a real numbers converging to$a$. The Aitkendelta – squared process consists of $\{a_n\}$transforming into the new sequence $\{b_n\}$ defined , by the formula

$$b_n = \frac{a_n a_{n+2} - a_{n+1}^2}{a_{n+2} - 2a_{n+1} + a_n} \quad (1)$$

Symbolized with the symbol At in this paper.

Richardson Extrapolation

Let $Q(k)$ be an approximation for the value of the integral$J = \int_{t_0}^{t_{2s}} \tau(t)dt$ that depends on a positive distinct step size k with acorrection terms $C_Q(k) = \sum_{i=0}^{\infty} \omega_i k^{r_i}$, where the $\omega_i$are unknown constants and the r_i are known constants. The formula of Richardson extrapolation is

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$$Q\left(k,\frac{k}{2}\right)=\frac{2^{r_i}Q\left(\frac{k}{2}\right)-Q(k)}{2^{r_i}-1} \tag{2}$$

The symbol Re will be used to express for it here.

Newton-Cotes rules

Broadly, the Newton-Cotes methods can be formed as follows

$$J=\int_{t_0}^{t_{2s}}\tau(t)dt=Q(k)+C_Q(k)+R_Q \tag{3}$$

by assuming that the number of partial periods is $2s$,where $Q(k)$is the expression of approximation for the value of the integral J by using the numerical rule$Q$, $C_Q(k)$is the error formula for $Q(k)$and R_Q is the remainder which is related after using a several terms of $C_Q(k)$ and $k=\frac{(t_{2s}-t_0)}{2s}$, Syfi [10].

Number of the partial periods was be chosen an even integer on account of the Simpson's rule applies only if the number of periods of fragmentation is even and this choice has no effect on the rest of the rules.

The formulas of$Q(k)$can be written as follows:

$$Q(k)=k\left(\lambda\tau_0+\bar{\lambda}\tau_1+\bar{\bar{\lambda}}\tau_2+\cdots+\bar{\lambda}\tau_{2s-1}+\lambda\tau_{2s}\right) \tag{4}$$

where$\tau_i=\tau(t_i)$,$t_i=t_0+ik$,$i=0,1,2,\cdots,2s$ and the wattage transactions take the arrangement $\lambda,\bar{\lambda},\bar{\bar{\lambda}},\cdots,\bar{\lambda},\lambda$.

So $2s+1$ of a pivotal points were needed , the equation(4)can be simplified by taken

$$\bar{\lambda}=2(1-\lambda)\text{ and }\bar{\bar{\lambda}}=2\lambda \tag{5}$$

When $\lambda=\frac{1}{2}$obtained Trapezoidal rule which is denoted by $T(k)$:

$$T(k)=\frac{k}{2}\left[\tau_0+\tau_{2p}+2\sum_{i=1}^{2s-1}\tau_i\right] \tag{6}$$

When $\lambda=\frac{1}{3}$obtained Simpson′s rule which is denoted by $S(k)$:

$$S(k)=\frac{k}{3}\left[\tau_0+\tau_{2p}+4\sum_{i=1}^{2s-1}\tau_{2i-1}+2\sum_{i=1}^{2s}\tau_{2i}\right] \tag{7}$$

If $\lambda=0$obtainMid-point method which is denoted by $M(k)$:

$$M(k)=k\sum_{i=1}^{2s-1}\tau_{i+\frac{k}{2}} \tag{8}$$

Before finding the correction terms$C_Q(k)$define theoperators E and D as follow:

$$E\tau(k)=\tau(k+1)\quad\text{(shfting operator)}$$
$$D\tau(k)=\tau'(k)$$

The correction terms$C_Q(k)$ can be written by the formula

$$C_Q(k)=D^{-1}(E^{2s}-1)\tau_0-Q(k) \tag{9}$$

From the equations (4) and (5) obtain

$$k^{-1}Q(k)=\left[\frac{\lambda(E-1)^2+2E}{E^2-1}\right](E^{2s}-1)\tau_0 \tag{10}$$

The equations (9) and (10) lead to

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$$C_Q(k) = k\left[\frac{(kD)^{-1} + \lambda(E-1)^2 + 2E}{E^2 - 1}\right](\tau_{2s} - \tau_0) \qquad (11)$$

Since

$$\tanh\left(\frac{1}{2}kD\right) = \frac{E-1}{E+1} = \frac{(E-1)^2}{E^2-1} \text{ and } \operatorname{sech}(kD) = \frac{2E}{E^2-1} \text{ where } E = e^{kD}.$$

Then, the equation (11) becomes:

$$C_Q(k) = k\left[(kD)^{-1} + \tanh\left(\frac{1}{2}kD\right) + \operatorname{sech}(kD)\right](\tau_{2s} - \tau_0) \qquad (12)$$

By using Taylor's series of $\tanh\left(\frac{1}{2}kD\right)$ and $\operatorname{sech}(kD)$ obtain:

The correction terms of Trapezoidal, Mid-point and Simpson rules are respectively:

$$C_T(k) = -\frac{1}{12}k^2(\tau_{2s}' - \tau_0') + \frac{1}{720}k^4(\tau_{2s}''' - \tau_0''') - \frac{1}{30240}k^6\left(\tau_{2s}^{(5)} - \tau_0^{(5)}\right) + \cdots \qquad (13)$$

$$C_M(k) = \frac{1}{6}k^2(\tau_{2s}' - \tau_0') - \frac{1}{360}k^4(\tau_{2s}''' - \tau_0''') + \frac{1}{15120}k^6\left(\tau_{2s}^{(5)} - \tau_0^{(5)}\right) + \cdots \qquad (14)$$

$$C_S(k) = -\frac{1}{180}k^4(\tau_{2s}''' - \tau_0''') + \frac{1}{1512}k^6\left(\tau_{2s}^{(5)} - \tau_0^{(5)}\right) + \cdots \qquad (15)$$

Fox[7]

If the function of integration is continous, then the correction terms can be written by the formulas:

$$J - T(k) = \omega_{T1}k^2 + \omega_{T2}k^4 + \omega_{T3}k^6 + \cdots$$
$$J - M(k) = \omega_{M1}k^2 + \omega_{M2}k^4 + \omega_{M3}k^6 + \cdots$$
$$J - S(k) = \omega_{S1}k^4 + \omega_{S2}k^6 + \omega_{S3}k^8 + \cdots$$

Where $\omega_{T1}, \omega_{T2}, \omega_{T3}, \cdots, \omega_{M1}, \omega_{M2}, \omega_{M3}, \cdots, \omega_{S1}, \omega_{S2}, \omega_{S3}, \cdots$ are constants.

A new Extrapolation Rule

Theorem: Let $Q(k)$ be an approximation value of the integral $J = \int_{t_0}^{t_m} \tau(t)dt$ which was be calculated by a numerical integral method with a correction terms take the formula $C_Q(k) = \sum_{i=0}^{\infty} \omega_i k^{r_i}$, where the ω_i are unknown constants and the r_i are known constants, then value of the integral can be calculated by:

$$J = \frac{\left[Q(k)(2^{r_1+1} - 2^{2r_2}) + Q\left(\frac{k}{2}\right)(2^{2r_2+1} - 2^{2r_1+1}) + Q\left(\frac{k}{4}\right)(2^{2r_1+2r_2} - 2^{r_1+2r_2+1})\right]}{2^{2r_2}(2^{2r_1} - 2^{r_1+1} + 1) + 2^{r_1+1}(1 - 2^{r_1})}$$

Proof

Suppose that the integral was be calculated by one of the Newton-Cotes rules, then

$$J = \int_{t_0}^{t_m} \tau(t)dt = Q(k) + C_Q(k) + R_Q.$$

Carelessly R_Q, the above formula was be written as follow:

$$J = Q(k) + \sum_{i=0}^{\infty} \omega_i k^{r_i}.$$

By taking three different numbers of divisions m_1, m_2 and m_3 suchthat $m_1 = 2m_2 = 4m_3$ into which the integration period is separated by finding the values of integration using one of the Newton-Cotes rules. Note that k is the distance between two consecutive nodes in the interval $[t_0, t_m]$.

$$\text{Then } k_i = \frac{t_m - t_0}{m_i} = \frac{k_{i-1}}{2}.$$

So the correction terms can be written as

$$J - Q(k) = \omega_1 k^{r_1} + \omega_2 k^{r_2} + \omega_3 k^{r_3} + \cdots \qquad (16)$$

$$J - Q\left(\frac{k}{2}\right) = \omega_1\left(\frac{k}{2}\right)^{r_1} + \omega_2\left(\frac{k}{2}\right)^{r_2} + \omega_3\left(\frac{k}{2}\right)^{r_3} + \cdots \qquad (17)$$

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$$J - Q\left(\frac{k}{4}\right) = \omega_1\left(\frac{k}{4}\right)^{r_1} + \omega_2\left(\frac{k}{4}\right)^{r_2} + \omega_3\left(\frac{k}{4}\right)^{r_3} + \cdots \qquad (18)$$

By adopting the first and second terms and neglecting the rest in three equations above deduce that:

$$J - Q\left(\frac{k}{4}\right) = \left(\frac{k}{4}\right)^{r_1}\frac{J(2^{r_1} - 2^{r_1+1}) - Q(k)2^{r_1} + Q\left(\frac{k}{2}\right)2^{r_1+1}}{k^{r_2}(2^{r_1} - 2)}$$

$$+\left(\frac{k}{4}\right)^{r_2}\frac{J(2^{r_1+1} - 2) - Q\left(\frac{k}{2}\right)2^{r_{1+1}} + 2Q(k)}{k^{r_2}(2^{r_1} - 2)}$$

$$= \frac{J(2^{r_1} - 2^{r_1+1})2^{2r_2} + J(2^{r_1+1} - 2)2^{2r_1} + Q\left(\frac{k}{2}\right)(2^{r_1+2r_2+1} - 2^{3r_1+1}) + Q(k)(2^{2r_1+1} - 2^{r_1+2r_2})}{2^{2r_1+2r_2}(2^{r_1} - 2)}$$

$$= \frac{J(2^{r_1+2r_2} - 2^{r_1+2r_2+1} + 2^{3r_1+1} - 2^{2r_1+1}) + Q\left(\frac{k}{2}\right)(2^{r_1+2r_2+1} - 2^{3r_1+1}) + Q(k)(2^{2r_1+1} - 2^{r_1+2r_2})}{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1})}$$

$$J - \frac{J(2^{r_1+2r_2} - 2^{r_1+2r_2+1} + 2^{3r_1+1} - 2^{2r_1+1})}{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1})}$$

$$= \frac{Q\left(\frac{k}{2}\right)(2^{r_1+2r_2+1} - 2^{3r_1+1}) + Q(k)(2^{2r_1+1} - 2^{r_1+2r_2})}{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1})} + Q\left(\frac{k}{4}\right)$$

$$J\frac{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1} - 2^{r_1+2r_2} + 2^{r_1+2r_2+1} - 2^{3r_1+1} + 2^{2r_1+1})}{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1})}$$

$$= \frac{Q\left(\frac{k}{2}\right)(2^{r_1+2r_2+1} - 2^{3r_1+1}) + Q(k)(2^{2r_1+1} - 2^{r_1+2r_2})}{(2^{3r_1+2r_2} - 2^{2r_1+2r_2+1})} + Q\left(\frac{k}{4}\right)$$

$$J = \frac{2^{r_1}\left[Q(k)(2^{r_1+1} - 2^{2r_2}) + Q\left(\frac{k}{2}\right)(2^{2r_2+1} - 2^{2r_1+1}) + Q\left(\frac{k}{4}\right)(2^{2r_1+2r_2} - 2^{r_1+2r_2+1})\right]}{2^{r_1}(2^{2r_1+2r_2} - 2^{r_1+2r_2+1} - 2^{2r_2} + 2^{2r_2+1} - 2^{2r_1+1} + 2^{r_1+1})}$$

$$J = \frac{\left[Q(k)(2^{r_1+1} - 2^{2r_2}) + Q\left(\frac{k}{2}\right)(2^{2r_2+1} - 2^{2r_1+1}) + Q\left(\frac{k}{4}\right)(2^{2r_1+2r_2} - 2^{r_1+2r_2+1})\right]}{2^{2r_2}(2^{2r_1} - 2^{r_1+1} + 1) + 2^{r_1+1}(1 - 2^{r_1})} \cdots (19)$$

Equation(19)willbe called theAl-karamy2accelerationand sign with the symbol Ka2.

Examples and Results

Applying the new acceleration formula (Ka2) to calculation a lot of continuous, impaired in derivative or impaired in one or both ends of integration and get good results in terms of accuracy and speed of approach, five of which are listed below, applied the Mid-point method (M) with the new acceleration formula and compared the results with the Richardson (Re) and Aitken (At) rules.

Table 1: Explain the Correct Number of Decimal Places on Number of Partial Periods m

No.	Integral	$\frac{M}{m}$	$\frac{At}{m}$	$\frac{Re}{m}$	$\frac{Ka2}{m}$
1	$\int_1^2 \frac{3\ln x}{x}dx$	$\frac{6}{512}$	$\frac{5}{85}$	$\frac{7}{1991}$	$\frac{15}{512}$
2	$\int_2^3 \frac{\sin x}{\sqrt{1+\cos x}}dx$	$\frac{5}{64}$	$\frac{4}{34}$	$\frac{9}{3827}$	$\frac{14}{64}$
3	$\int_0^1 x^{3/2}dx$	$\frac{6}{512}$	$\frac{6}{187}$	$\frac{8}{3528}$	$\frac{15}{512}$
4	$\int_5^6 \left(1-\frac{\sin x}{\ln x}\right)dx$		$\frac{5}{34}$	$\frac{8}{1544}$	$\frac{14}{128}$
5	$\int_2^3 \frac{e^x}{x^2}dx$		$\frac{5}{69}$	$\frac{8}{2570}$	$\frac{14}{256}$

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Example 1: The analytical value of this integral is 0.720679520877302rounded to fifteen decimal places. The function of integration is continuous for each point at $[1,2]$. Applying Mid-point rule withaccelerationsKa2,(Re) and (At)obtained the results listed in two tables (2) and (3).At$m = 512$ theapproximated value is correct for six decimal places by using Mid-point rule while it is equal to analytical value with applying the acceleration Ka2.

But by using the other accelerations(Re) at $m = 1991$ and (At) at $m = 85$the best results were seven and five decimal places respectively, if the number of periods of fragmentation increases, the numerical value of the integration fluctuates when the accelerators are used.

This meaning that the Al-karamy2acceleration is the bester than Richardson and Aitken.

$$\text{Table(2): Shows the values of} \int_1^2 \frac{3\ln x}{x}\,dx \text{ by usingMid} - \text{point rulewith Ka2}$$

m	M	Ka2: $r_i = (2,4)$	Ka2: $r_i = (4,6)$	Ka2: $r_i = (6,8)$	Ka2: $r_i = (8,10)$
1	0.810930216216329				
2	0.747442936950271				
4	0.727748053056221	0.722846459988431			
8	0.722473502426434	0.720867960558183			
16	0.721129757897945	0.720692766746376	0.720684725317614		
32	0.720792190049460	0.720680377373050	0.720679653050836		
64	0.720707695057103	0.720679574886841	0.720679523374961	0.720679522128099	
128	0.720686564852941	0.720679524260512	0.720679520918668	0.720679520886601	
256	0.720681281898134	0.720679521088872	0.720679520877958	0.720679520877350	0.720679520877331
512	0.720679961134192	0.720679520890527	0.720679520877312	0.720679520877302	0.720679520877302

$$\text{Table(3)compare the values of} \int_1^2 \frac{3\ln x}{x}\,dx \text{ by using Ka2, Re and At.}$$

Value by acceleration		m	**analytical value**
Ka2	0.720679520877302	512	0.720679520877302
Re	0. 7206795	1991	
At	0.720684	85	

Example 2: Hereisa continuous function for each point at$[2,3]$, the analytical value of this integral is 1.32813067755479 rounded to fourteen decimal places. Using Mid-point rule with accelerationsKa2 , (Re) and (At) obtained the results listed in two tables (4) and (5). When $m = 64$ the approximated value is correct for five decimal places by using Mid-point rule whilst it is equal to analytical value with applying the acceleration Ka2. But by applied the acceleration(Re) at $m = 3827$ get nine decimal places and with (At) at $m = 34$the best result be only five decimal places. So Al-karamy2accelerationis the best in terms of accuracy and speed of approach.

$$\text{Table(4)Shows the values of} \int_2^3 \frac{\sin x}{\sqrt{1+\cos x}}\,dx \text{ by usingMid} - \text{point rulewith Ka2}$$

m	M	Ka2: $r_i = (2,4)$	Ka2: $r_i = (4,6)$	Ka2: $r_i = (6,8)$
1	1.34206691917614			
2	1.33159566647222			
4	1.32899574018164	1.32812131572060		
8	1.32834686928820	1.32813009574751		
16	1.32818472086972	1.32813064124319	1.32813067783942	
32	1.32814418809490	1.32813067528611	1.32813067755921	
64	1.32813405517178	1.32813067741301	1.32813067755486	1.32813067755479

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Table(5)compare the values of $\int_2^3 \frac{\sin x}{\sqrt{1+\cos x}}dx$ by using Ka2, Re and At.

Value by acceleration		m	analytical value
Ka2	1.32813067755479	64	1.32813067755479
Re	1.328130678	3827	
At	1.32813	34	

Example 3: The analytical value of this integral is0.4,this is an improper integral in the derivative at lower limit of integration, so the correction terms is

$$C_Q(k) = \omega_1 k^2 + \omega_2 k^{5/2} + \omega_3 k^4 + \omega_4 k^6 + \cdots \text{, Fox[3].}$$

The approximation value of this integral is 0.399999764358862by using Mid-point method, while it is matching to the analytical value when applying theAl-karamy2acceleration at $m = 512$, this results written in table (6). However, the approximated values of this integral were minimize by using the other accelerations (Richardson and Aitken).

From the results in two tables (6) and (7) concluded that the Al-karamy2acceleration is the bester than Richardson and Aitken in terms of accuracy and speed of approach.

Table(6)Shows the values of $\int_0^1 x^{3/2}dx$ by usingMid − point rule with Ka2

m	M	Ka2: $r_i = (2,2.5)$	Ka2: $r_i = (2.5,4)$	Ka2: $r_i = (4,6)$	Ka2: $r_i = (6,8)$
1	53553390593274				
2	37259526419165				
4	96606818742048	0.399387761625526			
8	99114337804122	0.399888106225533			
16	99771941117509	0.399979970022746	0.399998837128331		
32	99941808515279	0.399996443137776	0.399999921695093		
64	99985243955453	0.399999370220699	0.399999994993683	0.399999999491068	
128	99996274179352	0.399999888606523	0.399999999685199	0.399999999990176	
256	99999062037193	0.399999980304279	0.399999999980294	0.399999999999835	0.399999999999958
512	99999764358862	0.399999996518009	0.399999999998768	0.399999999999998	0.400000000000000

Table(7)compare the values of $\int_0^1 x^{3/2}dx$ by using Ka2, Re and At.

Value by acceleration		m	analytical value
Ka2	0.4	512	0.4
Re	0.399999995	3528	
At	0.39999995	187	

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Example 4: The function of integration is continuous for each point in the interval $[5,6]$. This integral is unknown analytical value, the approximated value of integration can be predicted by proving the numerical values, the results were approaching and the results correspond horizontally to several columns to a appointed value when applying the Al-karamy2accelerationat $m = 128,256,512\ and\ 1024$.The approximation value of this integral can be expected 1.40114100975549 as described in table (8).

On the other hand the approximate value can't be predicted by using the other accelerations.

The above shows the superiority of Al-karamy2 formula on the Richardson and Aitken accelerations where the results in tables (8) and (9) illustrate this clearly.

Table(8)Shows the values of $\int_5^6 \left(1 - \frac{\sin x}{\ln x}\right) dx$ by usingMid − point rulewith Ka2

m	M	Ka2: $r_i = (2,4)$	Ka2: $r_i = (4,6)$	Ka2: $r_i = (6,8)$	Ka2: $r_i = (8,10)$	Ka2: $r_i = (10,12)$
1	1.41386779007565					
2	1.40428068617539					
4	1.40192336410248	40112042416988				
8	1.40133643899141	40113975072461				
16	1.40118985711979	40114093149114	1.40114101212500			
32	1.40115322097525	40114100487059	1.40114100979209			
64	1.40114406252160	40114100945029	1.40114100975606	40114100975548		
128	1.40114177294459	40114100973642	1.40114100975550	40114100975549		
256	1.40114120055262	40114100975430	1.40114100975549	40114100975549	.40114100975549	
512	1.40114105745476	40114100975542	1.40114100975549	40114100975549	.40114100975549	
1024	1.40114102168031	40114100975549	1.40114100975549	40114100975549	.40114100975549	1.40114100975549

Table(9)compare the values of $\int_5^6 \left(1 - \frac{\sin x}{\ln x}\right) dx$ by using Ka2, Re and At.

Value by acceleration		m	**analytical value**
Ka2	1.40114100975549	128	unknown
Re	1.40114101	1544	
At	1.40114	34	

Example 5: By using the fundamental calculation theorems can’t calculating this integral. The function of integration is continuous for each point in$[1,2]$.

The results in tables (10), accentuated that the values converge towards the value of 1.97894728969296 as well as matching the values of integration in the last three rows when $m = 256, 512$ and 1024.

While, the approximate value can't be foreseen by using the Richardson and Aitken accelerations.

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The above shows the priority of Al-karamy2acceleration on the Richardson and Aitken accelerations where the results in tables (10) and (11) explain this clearly.

Table(10)Shows the values of $\int_2^3 \frac{e^x}{x^2}dx$ by usingMid − point rule with Ka2

m	M	Ka2: $r_i = (2,4)$	Ka2: $r_i = (4,6)$	Ka2: $r_i = (6,8)$	Ka2: $r_i = (8,10)$	Ka2: $r_i = (10,12)$
1	1.94919903371256					
2	1.97128392516570					
4	1.97701553436737	1.97883453425578				
8	1.97846332082889	1.97893944040035				
16	1.97882623250297	1.97894678317580	1.97894721719862			
32	1.97891702132507	1.97894725776737	1.97894728830076			
64	1.97893972234644	1.97894728769334	1.97894728966976	1.97894728968734		
128	1.97894539784042	1.97894728956792	1.97894728969259	1.97894728969293		
256	1.97894681672883	1.97894728968515	1.97894728969296	1.97894728969296	1.97894728969296	
512	1.97894717145187	1.97894728969247	1.97894728969296	1.97894728969296	1.97894728969296	
1024	1.97894726013268	1.97894728969293	1.97894728969296	1.97894728969296	1.97894728969296	1.97894728969296

Table(11)compare the values of $\int_2^3 \frac{e^x}{x^2}dx$ by using Ka2, Re and At.

Value by acceleration		m	analytical value
Ka2	1.97894728969296	256	unknown
Re	1.978947285	2570	
At	1.97894	69	

II. Discussion and Conclusion

The results of this research show that when using Mid-point method to calculate the approximate values of numerical integrations, it gives precise values for several decimal places, this accuracy ranges from five to six correct decimal places and in periods ranging from 64 to 512 partial periods.

However, after using Al-karamy2 formula, the results are more accurate between fourteen and fifteen decimal places with the same number of partial periods. This method is clearly superior to the Richardson and Aitken accelerations in terms of accuracy and number of periods of fragmentation, the first accuracy ranges from seven to nine correct decimal periods and partial periods between 1991 and 3827, while the accuracy of the second is between four decimal places to six correct decimal places and ranging from 34 to 187 partial periods.

From the above we conclude that it is possible to rely on the Al-karamy2method to increase the accuracy of numerical integrals results, also see clearly the superiority of the new formula on the other accelerations in terms of precision and speed of approach. It should be noted that the new formula enables us to predict the values of integrations unknown analytical value without doubt, by approaching the value of numerical integration of a certain amount and its horizontal confirmation is the purpose of the series of correction limits where it is a series of Cauchy's convergent and this possibility is not available in the Richardson and Aitken as shown in integrations fourth and fifth.

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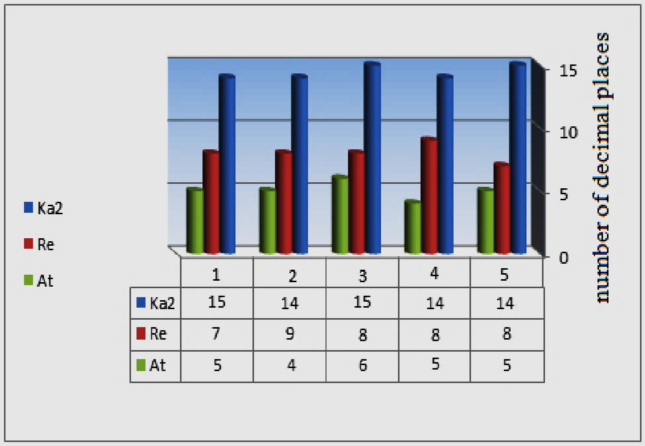

Chart showing the number of decimal places obtained by applying the three accelerators

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