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Extending Semimodules over Semirings

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Extending Semimodules over Semirings

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Abstract. The objective of our research paper is to introduce as well as to study many essential properties of the concept of extending semimodules. A semimodule S is named extending (CS) if every subsemimodule of S is essential in a direct summand of S. Therefore, extending semimodule behaviour with respect to direct sums and direct summands are examined. Moreover, studying some properties of these semimodules concepts, e.g., every direct summand of a CS-semimodule is a CS-semimodule. While the direct sum of extending semimodules is not necessarily extending.

1. Introduction
The A *T*-module *S* is called an extending module (*CS*–module) based on the extending property as follows: for each submodule *X* of *S*, there exists a direct summand *N* of *S*, which is an essential extension of *X*. It is known that a complement submodule need not be a summand, in the class of *CS*–modules any complement is a summand. Originality of *CS*-modules was presented by Von Neumann in 1930 [1]. In 1960, Utumi has studied this condition (identifying it as C_1 condition) in his study on self-injective and continuous ring [2]. In fact C_1 condition is common generalization of the injective and the semi simple condition, this motivates the name of extending condition. Another name of this condition is *CS* condition. it has developed in many articles and in at least [3][4][5].
In recent years, the extending modules theory has come to represent an important role and generally major contributions to this theory, through its widely available interesting findings on expanding properties in the theoretical preparation of the module. For background and applications of extending module (see [6]).

In this work, the extending semimodule over a semiring will be introduced and investigated. A semiring can be defined as a set *T*, which is non-empty together with two binary operations multiplication (.) and addition (+) ; as mentioned that $(T, .)$ is a monoid with an identity element $1 \neq 0$; $(T, +)$ is a commutative monoid with identity element 0; $t0 = 0t = 0$ for all $t \in T$; $a_1(a_2 + a_3) = a_1a_2 + a_1a_3$ and $(a_2 + a_3)a_1 = a_2a_1 + a_3a_1$; for all $a_1, a_2, a_3 \in T$. The semiring *T* is commutative if the monoid $(T, .)$ is commutative [7]. Let $(S, +)$ be an additive abelian monoid with additive identity 0_S. Then *S* is named a left T-semimodule if there exists a scalar multiplication $T \times S \rightarrow S$ defined by $(t, x) \mapsto tx$, such that $t(x + y) = tx + ty$; $(ts)x = t(sx)$; $(t + s)x = tx + sy$; $0_T S = t0_S = 0_S$ for all $x, y \in S$ and for all $t, s \in T$ [7].

A non-empty subset K of a left–semimodule S is called subsemimodule of S if K is closed under addition and scalar multiplication, that is K a T-semimodule itself (denoted by $K \leq S$) [8]. A T-semimodule S is said to be a direct sum of subsemimodules $S_1, S_2, \ldots, S_k$ of S, if each $s \in S$ can be written uniquely as $s = s_1 + s_2 + \ldots + s_k$, where $s_i \in S_i$. It is denoted by $S = S_1 \oplus S_2 \oplus \ldots \oplus S_k$. In this case each S_i is called **a** direct summand of S [9].

If T is a semiring and S, N are left T-semimodules, then a map $P{:}S \rightarrow N$ is called a **homomorphism** of T-semimodules, if satisfied the following, $P(s + s') = P(s) + P(s')$; $P(t\, s) = tP(s)$, for all $s, s' \in S$ and $t \in T$. The set of T-homomorphism's of S into N is denoted by $Hom(S, N)$. A homomorphism P is called an **epimorphism** if it's onto, it is called a **monomorphism** if P is one-one, and it is isomorphism if P is one-one and onto, and $ker(P) = \{s \in S | P(s) = 0\}$. If P is a homomorphism from a T-semimodule to itself, P is called endomorphism of S, $End(S)$ means the set of all T- endomorphism's of S. Using standard arguments, it can be shown that for each T-semimodule S, $End(S)$ is a semiring [7].

A **subtractive** subsemimodule (or k-subsemimodule) K is a subsemimodule of S such that if $k, k + s \in K$ then $s \in K$ [9]. Note that any direct summand is subtractive [11]. A semimodule S is said to be **semi subtractive**, if for any $s, s' \in S$ there is always some $h \in$ Satisfying $s + h = s'$ or $s' + h = s$ [7]. A nonzero T-subsemimodule K of S is named **essential** (large) and write ($K \leq^e S$), if $K \cap L \neq 0$ for every nonzero subsemimodule L of S [12]. A subsemimodule K of semimodule S is called **closed** if K has no proper essential extension in S (denoted by $N \leq^c S$) [13].

Let S be a T-semimodule, A and B are subsemimodules of S; A is called **intersection complement** (briefly complement) of B if $A \cap B = 0$ and A is maximal in the set of all subsemimodules of S that have zero intersection with B [13]. A subsemimodule K of a semimodule S is said to be **closure** of a subsemimodule N in S if K is closed and N essential in K [13]. A T-semimodule N is named (**A-injective or injective relative to A**), if for any subsemimodule V of A, each homomorphism from V into N can be extended to a T-homomorphism from A to N. The T-semimodule N is injective if it is injective relative to every T-semimodule [14]. If E is an injective T-semimodule, and it is a minimal injective extension of the T-semimodule S, then E is called an **injective hull** of S, denoted by $E(S)$.

2. CS-Semimodule

In this section, CS-semimodule will be presented as well as investigating some properties of them. Initially for this purpose, some properties of complement subsemimodules that are useful in analyzing the structure of extending semimodule, will be given.
According to [6], the concepts for modules will be converted for semimodule in the following.

Detention 2.1: A T-semimodule S is said to be **extending** (CS-semimodule) if every subsemimodule of S is essential in a direct summand of S.
It is clear that any simple T-semimodule (has no nontrivial subsemimodules) is CS. In fact any semisimple T- semimodule (has each subsemimodule as a summand) is CS, too.
It is known that any summand of a T-semimodule is closed, but the converse is not.

Example 2.2: $S = \mathbb{Z}_8 \oplus \mathbb{Z}_2$, $R = \mathbb{Z}$. Let $A = \langle(\bar{2}, \bar{1})\rangle$, then $A \leq^c S$, but not a summand of S.
Proposition 2.3: S is CS-semimodule if and only if every closed subsemimodule of S is a direct summand of S.
Proof: ($\Longrightarrow$) Assume S is a CS-semimodule, let $K \leq^c S$ then $K \leq^e S'$, where S' is a summand of S, then by definition of closed subsemimodule, $K = S'$, that is K is a summand of M.
($\Longleftarrow$) Conversely, let $K \leq S$, and let S' be closure of K, then $S'$is direct summand of S and $K \leq^e S'$, hence S is CS-semimodule. □
By (Proposition 2.3) S in example 2.2 is not CS.

Lemma 2.4: If $K\leq S$ *and* $L\leq^{e} S$ then $L\cap K \leq^{e} K$.
Proof: Clear. □
Lemma 2.5: If $K/L \leq^{e} K'/L$, then $K\leq^{e} K'$.
Lemma 2.6: If $L \leq^{c} S$ and $N\leq S$, then $L/N \leq^{c} S/N$
Proof: Assume that L/N is not closed in S/N then there exist $L'/N\leq S/N$ and $L/N\leq^{e} L'/N$ ($L\neq L'$), then $L\leq^{e} L'$, which contradicts the assumption $L \leq^{c} S$, hence $L/N \leq^{c} S/N$. □

Lemma 2.7: A subsemimodule K is closed in S if and only if whenever $K\leq N\leq^{e} S$ then $N/K\leq^{e} S/K$
Proof: By (Lemma 2.5), it is enough to prove the necessity condition.
Suppose $K \leq^{c} S$ and $K\leq N\leq^{e} S$. Let L be submodule of S such that $K\leq L$ and $(N/K)\cap(L/K)=0$, then $K=N\cap L\leq^{e} L$, based on Lemma(2.4). Since L is closed, then $L=K$ and $L/K=0$. Hence $N/K \leq^{e} S/K$. □

Lemma 2.8: If $K \leq^{c} N$ and $N \leq^{c} S$ then $K \leq^{c} S$.
Proof: Let K' be a complement of K in N and also N' be a complement of N in S, then $N \oplus N' \leq^{e} S$. Since N is closed in S (by assumption) and $N \leq N\oplus N' \leq^{e} S$, then based on (Lemma 2.7) $(N\oplus N')/N \leq^{e} S/N$. Sinc $(N\oplus N')/N \cong ((N\oplus N')/K)/(N/K)$ and, $S/N \cong (S/K)/(N/K)$, then $(N\oplus N')/K \leq^{e} S/K$. By the same way $(K\oplus K')/K \leq^{e} N/K$. Now, $(N\oplus N')/K = (N/K) \oplus((K + N')/K)$, then $(K + K' + N')/K = ((K + K')/K)\oplus((K + N')/K)) \leq^{e} S/K$.
Assume that $K \leq^{e} M \leq S$, then $K \cap (K' + N') = 0$ implies $M \cap (K' + N') = 0$. Therefore, $(M/K) \cap ((K + K' + N')/K) = K$. Thus $K=M$ and which implies $K \leq^{c} S$. □

Remark 2.9: Every subsemimodule K of a semimodule S is essential in closed subsemimodule H of S [13].

Proposition 2.10: Let $S = S_1\oplus S_2$ then S is *CS*-semimodule if and only if every complement of S_i, where (i = 1 or 2) is *CS*-semimodule and a direct summand of S.
Proof: Let K be complement of S_1 in S, since S is *CS*-semimodule, then K is closed subsemimodule of S, and by (Proposition 2.3), K is a direct summand of S.
Let L be closed subsemimodule of K, by (Lemma 2.8), $L \leq^{c} S$ and $L\cap S_1 = 0$, again since S is *CS*-semimodule, L is a direct summand of S, so $S = L\oplus L'$, for some $L'\leq S$ and by Modular Law $K = L\oplus(L'\cap K)$, therefore L is direct summand of K, and K is *CS*–semimodule.
Conversely, let $N \leq^{c} S$ then there exists *a* closed subsemimodule H of N such that $N\cap S_1\leq^{e} H$, clearly $(H\cap S_2) = 0$. By Zorn's Lemma, there exists a complement Q of S_2 in S with $H \leq Q$. Also, by (Lemma 2.8), $H \leq^{c} S$, hence $H \leq^{c} Q$, since Q is a complement of S_2 then by assumption, it is *CS*-semimodule, hence H is direct summand of S, therefore $S = H\oplus H'$ for some $H' \leq S$, by Modular Law $N = H\oplus(N\cap H')$, since $(N\cap H')$ is closed in S, and $(N\cap H') \cap S_1 = 0$ hence, $(N\cap H')$ is direct summand of S and also for H', $H' = (N\cap H') \oplus H''$ for some $H'' \leq S$, so $S = N\oplus H''$, therefore N is a direct summand of S. □

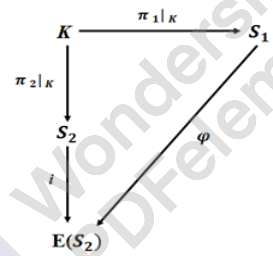
Example 2.11: $S = \mathbb{Z}_2\oplus \mathbb{Z}_4$. Let $N_i\leq S$, where (i = 1, 2, 3, 4,5) such that $N_1 = \mathbb{Z}_2 \oplus 0$, $N_2 = 0\oplus \mathbb{Z}_4$, $N_3 = \langle(\bar{1},\bar{1})\rangle = \langle(\bar{1},\bar{3})\rangle$, $N_4 = \langle(\bar{1},\bar{2})\rangle$, and $N_5 = \langle(0,\bar{2})\rangle$, N_2 and N_4 are complement of N_1, and they are direct summand of S, N_1 and N_3 are complement of N_2, and they are direct summand of S, then S is *CS*.
As it is mentioned before, the injective hull need not be exist for any semimodule. In the following results, the existence of injective hull is needed. For this purpose, we must add a condition that the semimodule has the injective hull.

Lemma 2.12: Let $S = S_1\oplus S_2$ be semimodule (with injective hull) and $\varphi \in Hom(S_1, E(S_2))$, $K = \{s_1+\varphi(s_1): s_1\in \varphi^{-1}(S_2)\}$ then:

1. $\varphi^{-1}(S_2) \cap K = Ker\varphi$.
2. If $\pi: S \rightarrow S_1$ is the natural projection then $\pi|_K$ is a monomorphism.

Proof: For (1), first we must prove that $K \cap S_2 = 0$, let $x \in K \cap S_2$, then $x = s + \varphi(s)$, $s \in \varphi^{-1}(S_2) \leq S_1$, $x \in S_2$ and $\varphi(s) \in S_2$ implies $s \in S_2$ (S_2 is subtractive since it is direct summand), so $s \in S_1 \cap S_2$, then $s = 0$ and $x = 0 + \varphi(0) = 0$.
Now, let $x \in \varphi^{-1}(S_2) \cap K$ then $x \in \varphi^{-1}(S_2)$ and $x + \varphi(x) \in K$, but $x \in K$, by subtractive property $\varphi(x) \in K$, hence $\varphi(x) \in K \cap S_2 = 0$, therefore $\varphi(x) = 0$ and $x \in ker\ \varphi$, so $\varphi^{-1}(S_2) \cap K \subseteq ker\ \varphi$, but $ker\ \varphi \subseteq \varphi^{-1}(S_2)$. On other hand, $x \in ker\ \varphi$, then $\varphi(x) = 0$ and $x + \varphi(x) \in K$, then $\varphi^{-1}(S_2) \cap K = ker\ \varphi$.
For (2), since $ker(\pi|_K) = ker\ \pi \cap K = S_2 \cap K = 0$, therefore $\pi|_K$ is monomorphism. □
In the next, we give a characterization of a complement subsemimodule, in certain cases.

Lemma 2.13: Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull) and K is a subsemimodule of S, then K is a complement of S_2 in S if and only if $K = \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$ for some $\varphi \in Hom(S_1, E(S_2))$.
Proof: Let K be a complement of S_2 in S, and $\pi_i: S \rightarrow S_i$, where ($i = 1, 2$) be the natural projections, since $ker(\pi_1|_K) = ker(\pi_1) \cap K = S_2 \cap K = 0$, then $\pi_1|_K$ is a monomorphism, consider the diagram as follow:



Where i the inclusion map, since $E(S_2)$ is injective, then there exists $\varphi \in Hom(S_1, E(S_2))$, such that $\varphi(\pi_1|_K) = i(\pi_2|_K)$. Let $x \in K$, then $x = \pi_1(x) + \pi_2(x)$, since $\varphi(\pi_1(x)) = i(\pi_2(x)) = \pi_2(x)$, then $x = \pi_1(x) + \varphi(\pi_1(x))$ and $x \in \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$, hence $K \subseteq \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$ and $\{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\} \cap S_2 = 0$ (note that S_2 is a summand of S, hence subtractive), since K is a complement of S_2 by assumption then $K = \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)$.
Conversely, suppose $\varphi \in Hom(S_1, E(S_2))$, and $K = \{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$, then $K \leq S$ and $K \cap S_2 = 0$. Now suppose $L \leq S$ and $L \cap S_2 = 0$, and $K \subseteq L$. Let $u \in L \backslash K$, then $u = \pi_1(u) + \pi_2(u)$ and $\pi_2(u) \neq \varphi(\pi_1(u))$, now $\varphi(\pi_1(u) \in E(S_2)$ and $S_2 \leq^e E(S_2)$ implies there exists $r \in T$ such that $0 \neq r\ \varphi(\pi_1(u) \in S_2$, therefore $\pi_1(ru) + \varphi(\pi_1(ru) \in K$, while $ru = \pi_1(ru) + \pi_2(ru)$.
Hence, $ru + \varphi(\pi_1(ru) = \pi_1(ru) + \varphi(\pi_1(ru) + \pi_2(ru)$, where $ru + \varphi(\pi_1(ru) \in L + S_2$, since $L + S_2$ is direct sum then $ru + \varphi(\pi_1(ru)$ [where $ru \in L$ and $\varphi(\pi_1(ru) \in S_2$], has unique representation then $\pi_2(ru) = 0$, hence $ru + \varphi(\pi_1(ru) \in L + 0$, therefore $\varphi(\pi_1(ru)) = 0$, but this is a contradiction, then $L = K$ and K is a complement of S_2 in S. □

Proposition 2.14: Any direct summand of a CS-semimodule is CS-semimodule.
Proof: Let C be a direct summand of S, then $S = C \oplus D$, for some $D \leq S$. Let $N \leq^c C$, then $N \cap D = 0$ and $C \leq^c S$, by (Lemma 2.8) $N \leq^c S$, therefore N is direct summand of S (since S is CS-semimodule), $S = N \oplus N'$ for some $N' \leq S$, by Modular Law $C = N \oplus (C \cap N')$, N is direct summand of C, hence C is CS-semimodule. □

Proposition 2.15: Let $S = S_1 \oplus S_2$ be a T-semimodule(with injective hull) then the following statements are equivalent:
1. S is CS-semimodule.
2. $\forall\ \varphi \in Hom(S_1, E(S_2))$, the subsemimodule $\{s_1 + \varphi(s_1): s_1 \in \varphi^{-1}(S_2)\}$ is CS-semimodule and a direct summand.

Proof: Suppose $\varphi: S_1 \rightarrow E(S_2)$ and let $K = \{s_1+\varphi(s_1): s_1 \in \varphi^{-1}(S_2)\} \leq S$, by (Lemma 2.13) K is a complement of S_2 in S, since S is CS-semimodule and by (Proposition 2.10) K is a CS-semimodule and direct summand of S.
Conversely, let $N \leq^c S$. If $N \cap S_1 = 0$ then by (Lemma 2.13), $N = \{x+\varphi(x): x \in \varphi^{-1}(S_1)\}$ for some $\varphi \in Hom(S_2, E(S_1))$ and by assumption it is a direct summand. If $N \cap S_1 \neq 0$, there exists a closed submodule K of N such that $N \cap S_1$ is essential in K. Clearly $K \cap S_2 = 0$. Let $\pi_i: S \rightarrow S_i$, where $(i = 1, 2)$ be the natural projections, then $\pi_1|K$ is a monomorphism and there exists $\varphi \in Hom(S_1, E(S_2))$ such that $\varphi(\pi_1(k)) = \pi_2(k)$ for all $k \in K$.
If $P = \{s_1+\varphi(s_1): s_1 \in \varphi^{-1}(S_2)\} \leq S$, then by (Lemma 2.13) P is a complement of S_2 in S, and it is a direct summand of S by assumption. Note that if $k \in K$, then $k = \pi_1(k) + \pi_2(k) = \pi_1(k) + \varphi(\pi_1(k)) \in P$, that is $K \leq P$. Since P is CS-semimodule (by assumption) K is a direct summand of P, hence K is a direct summand of S, say $S = K \oplus K'$, and by Modular Law we have $N = K \oplus N \cap K'$. Now, $N \cap K' \leq^c S$, clearly $(N \cap K') \cap S_1 = 0$ by an argument similar to the above $N \cap K'$ is a direct summand of S and hence also of K'. It follows that N is a direct summand of S. Thus S is CS-semimodule. □
For the next result the following lemmas are required.

Lemma 2.16: If $\alpha \in Hom(S, S')$ is an isomorphism and $N \leq^e S$, then $\alpha(N) \leq^e S'$.
Proof: Let $K \leq S' \ni \alpha(N) \cap K = 0$, then $\alpha^{-1}(\alpha(N) \cap K) = 0$, thus $N \cap \alpha^{-1}(K) = 0$, therefore $\alpha^{-1}(K) = 0$, and $K = 0$. □

Lemma 2.17: If $S \cong S'$, then S is CS-semimodule if and only if S' is CS-semimodule.
Proof: Immediately by definition and Lemma 2.16. □

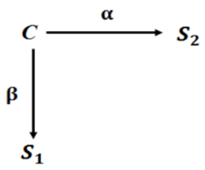
Proposition 2.18: If $S = S_1 \oplus S_2$ is a CS–semimodule (with injective hull), S_1 and S_2 are relative injective semimodules and $\varphi \in Hom(S_1, E(S_2))$, then $\varphi^{-1}(S_2)$ is CS–semimodule.
Proof: Let $N = \{x + \varphi(x): x \in \varphi^{-1}(S_2)\}$, by (Lemma 2.13) N is a complement of S_2, also by (Proposition 2.10) N is CS-semimodule, let $\pi_1|_N = \alpha$, then $\alpha: N \rightarrow S_1$ is a monomorphism. Let $y \in \alpha(N)$, then $y = \pi_1(n)$, for some $n \in N$ [since $n = x + \varphi(x)$, then $\pi_1(n) = \pi_1(x) \in \varphi^{-1}(S_2)$], $n = \pi_1(n) + \pi_2(n) = \pi_1(x) + \pi_2(n) = x + \varphi(x)$, then $\pi_1(x) = x$ and $\pi_2(n) = \varphi(x)$, hence $y \in \varphi^{-1}(S_2)$, therefore $\alpha(N) \subseteq \varphi^{-1}(S_2)$. If $x \in \varphi^{-1}(S_2)$, then $x \in S_1$ and $\varphi(x) \in S_2$, but $x + \varphi(x) \in N$, therefore $\alpha(x) + \alpha(\varphi(x)) \in \alpha(N)$, then $\alpha(x) = x \in \alpha(N)$, hence $\alpha(N) = \varphi^{-1}(S_2)$, hence $\alpha': N \rightarrow \varphi^{-1}(S_2)$ is an isomorphism. Since N is CS-semimodule by (Lemma 2.17), $\varphi^{-1}(S_2)$ is CS-semimodule. □

3. Direct Sum and Direct Summand of CS-Semimodule
The direct summand and direct sum of CS-semimodule will be studied in this section as well as conditions that ensure a subsemimodule of CS-semimodule to be CS-semimodule and supply related properties of CS-semimodule property.

Proposition 3.1: Let $S = S_1 \oplus S_2$ be a T-semimodule, where S_1 and S_2 are CS-semimodules, then S is CS-semimodule if and only if every closed $K \leq S$ with $K \cap S_1 = 0$ or $K \cap S_2 = 0$ is a direct summand.
Proof: ($\Longrightarrow$) It is proved by (Proposition 2.3).
($\Longleftarrow$) Let $B \leq^c S$, then either $B \cap S_1 = 0$, then by assumption B is direct summand of S. Or $B \cap S_1 \neq 0$, then there exists D such that $B \cap S_1 \leq^e D \leq^c B$ by (Remark 2.9), then $D \cap S_2 = 0$. Note that $D \leq^c S$ by (Lemma 2.8), then by assumption, D is a direct summand of S, that is, $S = D \oplus D'$ for some $D' \leq S$, by Modular Law $B = D \oplus (B \cap D')$, but $(B \cap D')$ is closed in S, then $(B \cap D') \cap S_2 = 0$, also by assumption $B \cap D'$ is a direct summand of D', then $D' = (B \cap D') \oplus D''$ for some $D'' \leq S$, so $S = D \oplus (B \cap D') \oplus D'' = B \oplus D''$, therefore B is a direct summand of S and S is CS-semimodule. □

<u>Lemma 3.2:</u> Let $S = S_1 \oplus S_2$, be a T-semisubtractive T-semimodule then S_1 is S_2-injective implies for every subsemimodule C of S with $C \cap S_1 = 0$, there exists a subsemimodule S' of S such that $S = S_1 \oplus S'$, $C \leq S'$.

***Proof*:** Assume that S_1 is S_2-injective, let $\pi_i: S \to S_i$, where (i = 1, 2) be the natural projections, let $C \leq S$ with $C \cap S_1 = 0$, consider the diagram where $\alpha = \pi_2|_C$ and $\beta = \pi_1|_C$, α is a monomorphism, by assumption there exists $f: S_2 \to S_1 \ni f\alpha = \beta$.



Define $S' = \{f(a) + a : a \in S_2\}$, then $S' \leq S$. For $c \in C$, $c = \pi_1(c) + \pi_2(c) = f(\pi_2(c)) + \pi_2(c) \in S'$, so $C \leq S'$. For $a \in S$, $a = \pi_1(a) + \pi_2(a)$, if $\pi_1(a) = f(\pi_2(a))$, then $a \in C \leq S' \leq S_1 \oplus S'$. If $\pi_1(a) \neq f(\pi_2(a))$, then by semisubtractive property either $\pi_1(a) + h = f(\pi_2(a))$ or $\pi_1(a) = f(\pi_2(a)) + h$ for some $h \in S$ (in any case $h \in S_1$, since S_1 is a direct summand, hence subtractive). So, either $a + h = \pi_1(a) + h + \pi_2(a) = f(\pi_2(a)) + \pi_2(a) \in S' \leq S_1 + S'$, hence $a \in S_1 + S'$. Or $a = \pi_1(a) + \pi_2(a) = f(\pi_2(a)) + h + \pi_2(a) = h + f(\pi_2(a)) + \pi_2(a) \in S_1 + S'$. Therefore $S = S_1 + S'$. On other hand, if $a \in S_1 \cap S'$ then $a \in S_1$ and $a = f(b) + b$, for $b \in S_2$, therefore $0 = \pi_2(a) = \pi_2(f(b)) + \pi_2(b) = 0 + \pi_2(b) = \pi_2(b)$ and $\pi_1(a) = \pi_1(f(b)) + \pi_1(b) = 0$, then $a = 0$, hence $S_1 \cap S' = 0$ and $S = S_1 \oplus S'$. □

For the following proposition, we give a condition of the direct sum of CS-semimodules to be CS-semimodule. □

<u>Proposition 3.3:</u> Let $S = S_1 \oplus S_2$ be a T-semimodule, where S_1 and S_2 are relative injective semimodules then S is CS-semimodule if and only if S_1 and S_2 are CS-semimodule.

***Proof*:** ($\Longrightarrow$) It is proved by (Proposition 2.14).

($\Longleftarrow$) Assume that S_1 and S_2 are CS-semimodule and S_i is S_j injective for (i, j=1, 2 and $i \neq j$), let $K \leq^c S$ and $K \cap S_1 = 0$, by (Lemma 3.2) there exists $S' \leq S$ such that $S = S_1 \oplus S'$ and $K \leq S'$, it is clear that $S' \cong S_2$ and hence S' is CS-semimodule by (Lemma2.17). On the other hand, $K \leq^c S'$(since it is closed in S) hence K is a direct summand of S', therefore K is a direct summand of S, similarly for any subsemimodule H of S with $H \cap S_2 = 0$, is a direct summand of S, therefore by (Proposition 3.1), S is CS-semimodule. □

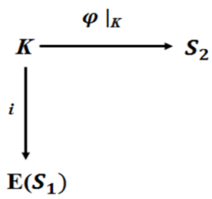
<u>Proposition 3.4:</u> Let $S = S_1 \oplus S_2$ be a T-semimodule, if S_1 is CS-semimodule and S_2 is S_1 injective then every closed subsemimodule K of S with $K \cap S_2 = 0$ is a direct summand of S.

***Proof*:** Let $K \leq^c S$ with $K \cap S_2 = 0$, since S_2 is S_1 injective by (Lemma 3.2) there exists $S' \leq S$ such that $K \subseteq S'$ and $S = S' \oplus S_2$, therefore $S' \cong S_1$, since S_1 is CS-semimodule, then S' is CS-semimodule and K is a direct summand of S' (say $S' = K \oplus K'$) hence $S = (K \oplus K') \oplus S_2 = K \oplus (K' \oplus S_2)$, that is, K is a direct summand of S, hence S is CS-semimodule. □

For determining under which condition a subsemimodule has a unique complement see the following.

<u>Lemma 3.5.</u> Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), and $\varphi \in Hom(S_1, E(S_2))$. If $Hom(S_2, E(S_1)) = 0$, then S_2 is a unique complement of $N = \{x + \varphi(x) : x \in \varphi^{-1}(S_2)\}$.

***Proof*:** Let $Y \leq S$, with $Y \cap N = 0$. Note that $ker(\varphi) \subseteq N$. Let $Y \cap \varphi^{-1}(S_2) = K$, if $K \neq 0$, then $\varphi|_K$ is a monomorphism [since $ker(\varphi|_K) = ker\, \varphi \cap K \subseteq N \cap K \subseteq N \cap Y = 0$]. Consider the diagram:



Since $E(S_1)$ is injective, there exists $0 \neq \alpha \in Hom(S_2, E(S_1))$, but this contradicts the assumption, then $K = 0$, therefore $Y \cap \varphi^{-1}(S_2) = 0$, but $\varphi^{-1}(S_2) \leq^e S_1$, then $Y \cap S_1 = 0$, hence $\pi_2|_Y$ is a monomorphism and $\pi_1(Y) = 0$. Therefore, $Y \subseteq S_2$, and S_2 is a unique complement of N. □

For a specific purpose, we derive a new lemma from Proposition 2.13 that will be more generality as follows:

***Lemma* 3.6:** Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), and $A \leq S$ with $A \cap S_2 = 0$, then $A \leq^c S$ if and only if $A = \{x + \varphi(x) : x \in X\}$ where $X \leq^c \varphi^{-1}(S_2)$, for some $\varphi \in Hom(S_1, E(S_2))$.

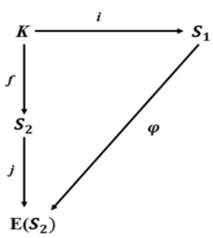
***Proof*:** ($\Longrightarrow$) Let $\pi_i : S \rightarrow S_i$, where ($i = 1, 2$) be the natural projections, since $A \cap S_2 = 0$, then $\pi_1{}' = \pi_1|_A : A \rightarrow S_1$ is a monomorphism, hence there exists $\varphi \in$ Hom(S_1, $E(S_2)$) such that $\varphi(\pi_1{}'(a)) = \pi_2{}'(a)$ for all $a \in A$, where $\pi_2{}' = \pi_2|_A$, then $\varphi(\pi_1(a)) = \pi_2(a)$.
Hence, for each $a \in A$, $a = \pi_1(a) + \pi_2(a) = \pi_1(a) + \varphi(\pi_1(a))$, so $A = \{x + \varphi(x) : x \in \pi_1(A)\}$, note that $\pi_2(A) = \varphi(\pi_1(A)) \subseteq S_2$, hence $\pi_1(A) \leq \varphi^{-1}(\pi_2(A)) \leq \varphi^{-1}(S_2)$, if $\pi_1(A) \leq^e Y \leq \varphi^{-1}(S_2)$, then $A + S_2 \leq^e \pi_1{}^{-1}(Y)$, but $S_2 \leq \pi_1{}^{-1}(Y)$, therefore $A \leq^e \pi_1{}^{-1}(Y)$, since A is closed in S, then $A = \pi_1{}^{-1}(Y)$, and $\pi_1(A) = Y$, thus $\pi_1(A) \leq^c \varphi^{-1}(S_2)$.
($\Longleftarrow$) if $A = \{x + \varphi(x) : x \in X\}$ and $X \leq^c \varphi^{-1}(S_2)$, it is clear that $A \leq N = \{x + \varphi(x) : x \in \varphi^{-1}(S_2)\}$, and that A has a proper essential extension in N if and only if X has a proper essential extension in $\varphi^{-1}(S_2)$, since X is closed in $\varphi^{-1}(S_2)$, it follows that $A \leq^c N$, then $A \leq^c S$. □

***Lemma* 3.7:** Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), where S_1 and S_2 are subsemimodules of S. If S_2 is S_1-injective then any closed subsemimodule A in S with $A \cap S_2 = 0$ must have the form $A = \{x + \varphi(x) : x \in X\}$, where X is closed subsemimodule of S_1 and $\varphi \in Hom(S_1, E(S_2))$.

***Proof*:** Let A be a closed subsemimodule in S with $A \cap S_2 = 0$, then by (Lemma 2.13) $A = \{x + \varphi(x) : x \in X\}$, where X is closed subsemimodule of $\varphi^{-1}(S_2)$, for some $\varphi \in Hom(S_1, E(S_2))$. But $\varphi^{-1}(S_2) \leq^e S_1$, so $X \leq^c S_1$. □

***Lemma* 3.8:** Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), where S_1 and S_2 are subsemimodules of S. If $\varphi^{-1}(S_2) = S_1$ for each $\varphi \in Hom(S_1, E(S_2))$, then S_2 is S_1-injective.

***Proof*:** Consider the diagram below, assume K is a subsemimodule of S_1 in S where i is the inclusion map, f is any homomorphism j is the inclusion map.



Since $E(S_2)$ is injective there exists $0 \neq \varphi \in Hom(S_1, E(S_2))$ such that $\varphi i = j$. Since $\varphi^{-1}(S_2) = S_1$, then $\varphi(S_1) \subseteq S_2$ and $\varphi \in Hom(S_1, S_2)$, therefore S_2 is S_1-injective. □

In the following, the condition of Proposition 3.4 gives extra results.

Proposition 3.9: Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), and $Hom(S_2, E(S_1)) = 0$, then S_1 is CS-semimodule and S_2 is S_1–injective if and only if every closed subsemimodule K of S with $K \cap S_2 = 0$ is a direct summand of S.

Proof: (⇒) It is proved by (Proposition 3.4).

(⇐) Suppose $K \leq^c S_1$, then $K \cap S_2 = 0$ and $K \leq^c S$ by (Lemma 2.8). By assumption K is a direct summand of S, say $S = K \oplus K'$, hence $S_1 = K \oplus K' \cap S_1$, therefore K is a direct summand of S_1, and S_1 is CS- semimodule.

Now, let $\alpha \in Hom(S_1, E(S_2))$ be arbitrary, then by (Lemma 2.13) $L=\{x+\alpha(x): x \in \alpha^{-1}(S_2)\}$ is closed in S and it is a complement of S_2, by assumption it is a direct summand of S If π_1 is the natural projection of $S = S_1 \oplus S_2$ onto S_1, then $y \in L$ implies $y = x+\alpha(x)$ for some $x \in \alpha^{-1}(S_2)$ and $\pi_1(y) = \pi_1(x) = x$, that is, $\pi_1(L) \subseteq \alpha^{-1}(S_2)$. If $x \in \alpha^{-1}(S_2)$, then $\pi_1(x) = x$ and $x+\alpha(x) \in L$ hence $x = \pi_1(x) = \pi_1(x+\alpha(x)) \in \pi_1(L)$, that is, $\pi_1(L) = \alpha^{-1}(S_2)$. Since $\pi_1(L)$ is closed in S_1 and $\alpha^{-1}(S_2)$ is essential in S_1, it follows $S_1 = \alpha^{-1}(S_2)$. Therefore, by (Lemma 3.8), S_2 is S_1-injective. □

Corollary 3.10: Let $S = S_1 \oplus S_2$ be a T-semimodule (with injective hull), and $Hom(S_2, E(S_1)) = 0$, then S is CS if and only if S_1 and S_2 are CS-semimodule and S_2 is S_1–injective **.**

Proof: (⇒) By (Propositions 2.14), both S_1 and S_2 are CS- semimodules, then by (Proposition 3.4**)** and (Proposition 3.9) S_2 is S_1–injective.

(⇐) This is proved by (Propositions 3.9, 2.13 and 2.15). □

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