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Onesided and Co-Onesided Approximation

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Abstract

Since shape constrains limit of the degree of approximation, we will relax the constraints of shape in small parts of the interval $I=[-1,1]$ and approximate a function f in the L_p^k space, $0<p\le\infty$, $k=0,1,or\,2$, which contains all functions $f\in L_p$ with $f^{(k)}\in L_p$, by an intertwining (co-onesided) pair of splines and/or polynomials, to get global estimates in terms of Ditzian-Totik modulus of smoothness.

We begin with improving Whitney's Theorem for onesided approximation by using Ditzian-Totik modulus of smoothness instead of $\tau-$modulus, to get less degree of approximation of the function $f\in L_p(I)$ conditioning that $\omega_r^{\varphi}(f)>0$.

Keywords: Approximation theories, constraints relaxation, splines

1. Introduction and Main Results

Throughout this article, we use the following notations from (Hu et. al., 1997);

Let $Y_s:=\{y_1,\cdots,y_s \mid y_0:=-1<y_1<y_2<\cdots<y_s<1=:y_{s+1}\}, s\ge 0$. We denote by $\Delta^0(Y_s)$ the set of all functions f such that $(-1)^{s-k}f(x)\ge 0$ for $x\in[y_k,y_{k+1}], k=0,\cdots s$.

That is, those that have $0\le s\le\infty$ sign changes at the points in Y_s and one nonnegative near 1. In particular, $\Delta^0=\Delta^0(Y_0)$ denotes the set of all nonnegative functions on $[-1,1]$. Functions f and g which belong to the same class $\Delta^0(Y_s)$ are said to be copositive.

Copositive approximation is the approximation of functions f from $\Delta^0(Y_s)$ class by polynomials and/or splines that are copositive with f. For $f\in L_p[-1,1]$, let $E_n(f)_p:=\inf_{p_n\in\Pi_n}\|f-P_n\|_p$, denotes the degree of unconstrained approximation.

The best onesided approximation of f by means of algebraic poly-nomials $P_n\in\Pi_n$ in L_p-- metric is given by $\widetilde{E}_n(f)_p:=\inf\left\{\|P-Q\|_p : P,Q\in\Pi_n \text{ and } P(x)\ge f(x)\ge Q(x), -1\le x\le 1\right\}$.

A natural extension of (co)positive and onesided approximations is the concept of so-called intertwining (co-onesided) approximation.

Definition 1.1. For the set $Y_s := \{y_1, \cdots, y_s \mid y_0 := -1 < y_1 < y_2 < \cdots < y_s < 1 =: y_{s+1}\}, s \geq 0$, the best intertwining polynomial approxima-tion of a function $f \in L_p[-1,1]$ is given by $\widetilde{E}_n(f)_p := \inf\{\|P-Q\|_p : P, Q \in \Pi_n \text{ and } P(x) \geq f(x) \geq Q(x), -1 \leq x \leq 1\}$.

We call $\{P,Q\}$ an intertwining pair of polynomials for f with respect to Y_s if $P-f, f-Q \in \Delta^0(Y_s)$.

Clearly, in the case $s=0$, the above definition becomes the definition of the best onesided polynomial approximation $\widetilde{E}_n(f, Y_0)_p = \widetilde{E}_n(f)_p$

Denote by $\omega_r(f,t)_p := \sup_{0<h\leq t}\|\Delta_h^r f\|_{L_p(I)}$, the classical modulus of smoothness, where $\Delta_h^r(f) := \sum_{i=0}^{r}(-1)^i\binom{r}{i} f\left(x + r\left(\frac{h}{2}\right) - ir\right)$.

The Ditzian-Totik modulus of smoothness which is defined for such an f as follows $\omega_r^{\varphi}(f,t)_p := \sup_{0<h\leq t}\|\Delta_h^r f\|_{L_p(I)}$.

There were other attempts made, the most notable being the works of Sendov and Popov, who gave the so called $\tau-$modulus, an averaged modulus of smoothness, defined for bounded measurable functions on $[a,b]$ by $\tau_r(f,t,J)_p := \|\omega_r(f,\cdot,t)\|_{L_p(I)}$, where $\omega_r(f,x,t)_p := \sup\left\{\left|\Delta_h^r(f,y)\right| : y \pm \frac{rh}{2} \in \left[x - \frac{rt}{2}, x + \frac{rt}{2}\right] \cap J\right\}$ is the *r*-th local modulus of smoothness of *f*.

Throughout this article, we use the following notations from (Bhaya, 2003)

Let $x_{-1} = 1, x_{n+1} = -1$ and for each $j = 0,1,\cdots,n$ set $x_j := x_{j,n} := \cos(j\pi/n)$, $I_j := [x_j, x_{j-1}]$, $h_j := |I_j| := x_{j-1} - x_j$ and $\Delta_n(x) := \sqrt{1-x^2}/n + 1/n^2$.

At first, we have to give some estimates which formulate the relations between the above measures as follows $\omega_r^{\varphi}(f,t)_p \leq \omega_r(f,t)_p \leq \tau_r(f,t)_p, 1 \leq p \leq \infty$, and $\omega_r^{\varphi}(f,t)_p \leq \omega_r(f,t)_p$, $0 \leq p \leq \infty$.

First of all, we'll introduce a proof for Whitney's theorem for onesided approximation in $L_p[-1,1]$ in Theorem I, which includes an equivalence between onesided approximation and Ditzian-Totik modulus of smoothness, taking into consideration the counter example for this theorem, whereas the authors in (Hu et. al., 1997), considered that only $\tau-$modulus is the correct modulus for the equivalence between onesided approximation and some modulus, which was proved in (Hu, 1995) by Hu, for a continuous function. It was stated in (Hu et. al., 1997) that "We also remark that τ is the "correct" modulus in $\widetilde{E}_n(f)_p \leq C(r)\tau_r(f,n^{-1})_p$, $1 \leq p \leq \infty$, i.e., it can't be replaced by ω or ω^{φ}, since the estimate $\widetilde{E}_n(f)_p \leq c\|f\|_p$, certainly cannot be correct for all $f \in L_p[-1,1]$, $p \leq \infty$. To see this, it is sufficient to consider the function f such that $f(0)=1$ and $f(x)=0$, $x \neq 0$, then $\|f\|_p = 0$ and $\widetilde{E}_n(f) > 0$". We can avoid the above counter example simply by assuming that $\|f\|_p > 0$, or more generally $\omega_r^{\varphi}(f,t) > 0$.

Theorem I. (Whitney's Theorem for Onesided Approximation)

For $f \in L_p[-1,1]$, $\omega_r^{\varphi}(f,|I|) > 0$, we have

$$\widetilde{E}_{r-1}(f)_p \sim w_r^{\varphi}(f,|I|)_p \tag{1}$$

Now, we get the way to approximate a differentiable function in $L_p[-1,1]$, by an intertwining pair of splines by dividing the interval $[-1,1]$, into small subintervals, each one contains at least four knots.

The following theorem is proved in (Hu et. al., 1997), by Hu, Kopotun and Yu in the space $W_p^1[-1,1]$ for the case $1 \leq p \leq \infty$ in terms of $\tau-$modulus of smoothness.

Theorem II. (Intertwining Spline Approximations, $0 \le p \le \infty$)

Let $f \in L_p^1[-1,1]$ and let $Y_s = \{y_1, \cdots, y_s \mid y_0 = -1 < y_1 < y_2 < \cdots < y_s < 1 = y_{s+1}\}, s \ge 0$ and let $r \ge 2$ be an integer. Let T_n be a given knot sequence such that there are at least $4(r-1)^2$ knots in each open intervals (y_i, y_{i+1}), $j = 1,2,\cdots,s-1$, then there exists an intertwining pair of splines $\{\overline{S}, S\}$ of order r on the knot sequence T_n, (i.e. $\overline{S}, S \in C^{r-2}[-1,1]$ and $\overline{S} - f$, $f - S \in \Delta^0(Y_s)$) such that, for $i = 1,2,\cdots,n-1$

$$\|\overline{S} - S\|_{Lp(I_i)} \le C|I_i|^2 w_{r-1}^{\varphi}(f', |I_i|, \mathrm{I}_i)_p \tag{2}$$

where C is a constant depending on r and on the maximum ratio $\rho := \max_{i=0}^{n-1} \frac{|I_{i\mp1}|}{|I_i|}$ and $\boldsymbol{I}_i$ is an interval such that $I_i \subseteq \boldsymbol{I}_i \subseteq \lfloor z_{i-6(r-1)^2}, z_{i+6(r-1)^2} \rfloor$.

Consequently, if in addition $f \in L_p^2[-1,1]$, then

$$\|\overline{S} - S\|_{Lp(I_i)} \le C|I_i|^3 w_{r-2}^{\varphi}(f'', |I_i|, I_i)_p \tag{3}$$

The proof of the above theorem also yields a more general result on onesided spline approximation.

Corollary III. Let $f \in L_p[-1,1]$, $0 \le p \le \infty$, and let $r \ge 2$ be an integer, then there exist splines $\overline{S}$ and S of order r on the knot sequence T_n, such that $\overline{S}(x) \ge f(x) \ge S(x), x \in [-1,1]$, and for $i = 1,2,\cdots,n-1$

$$\|\overline{S}_n - S_n\|_{Lp(I_i)} \le C w_r^{\vartheta}(f, |I_i|, I_i)_p \tag{4}$$

where C is a constant depending on r and on the maximum ratio $\rho := \max_{i=0}^{n-1} \frac{|I_{i\mp1}|}{|I_i|}$ and $\boldsymbol{I}_i$ is an interval such that $I_i \subseteq \boldsymbol{I}_i \subseteq \lfloor z_{i-6(r-1)^2}, z_{i+6(r-1)^2} \rfloor$.

The proof of the above corollary follows directly from the proof of theorem II which will be shown in Section 3, by omitting the inequality (15). Also, we'll use the previous theorem and its corollary to prove the following two theorems about onesided and co-onesided approximation, respectively, for large n.

Theorem IV. (Onesided Polynomial Approximation in $L_p[-1,1]$, $0 \le p \le \infty$)

Let $f \in L_p[-1,1]$ and $r \in N$. Then for every $n \ge r-1$, there exist polynomials $P, Q \in \Pi_n$, such that $P(x) \ge f(x) \ge Q(x), -1 \le x \le 1$ and

$$\|P - Q\|_p \le C(r) w_r^{\vartheta}(f, n^{-1})_p \tag{5}$$

Theorem V. (Intertwining Polynomial Approximation)

Let $f \in L_p^1[-1,1]$, $0 \le p \le \infty$, and let $Y_s = \{y_1, \cdots, y_s \mid y_0 = -1 < y_1 < y_2 < \cdots < y_s < 1 = y_{s+1}\}$, $s \ge 0$. Then

$$\widetilde{E}_n(f, Y_s)_p \le C(r,s) n^{-1} w_r^{\vartheta}(f', n^{-1})_p \tag{6}$$

Also, there exists an intertwining pair of polynomials $P, Q \in \Pi_n$ such that

$$\|P - Q\|_p \le C(r,s) n^{-1} w_r^{\vartheta}(f', n^{-1})_p \tag{7}$$

Moreover, if $f \in L_p^2[-1,1]$, then

$$\widetilde{E}_n(f, Y_s)_p \le C(r,s) n^{-1} w_r^{\vartheta}(f'', n^{-1})_p \tag{8}$$

2. Auxiliary Lemmas

We begin with some properties of D-T modulus of smoothness which are needed in the proofs of our main results. Note that the first lemma is valid for the range $0 \le p \le \infty$ which is proved by Ditzian and Totik (Ditzian and Totik, 1978) for $1 \le p \le \infty$ and by Ditzian, Hristov and Ivanov in their paper (Ditzian et. al., 1995) for the other cases

Lemma 2.1. For $f \in L_p$, $0 \le p \le \infty$, we have $\omega_r^{\varphi}(f,t)_p \le c\omega_r^{\varphi}(f,t)_p$, for $m \ge r$.

Another property which combining ω_r and ω_r^{φ} is proved by Petrushev and Popov for $1 \le p \le \infty$ in (Petrusher and Popov, 1987) and by Devore, Leviatan and Yu for $0 < p < 1$, in (DeVore et. al., 1992) is the next.

Lemma 2.2. For $f \in L_p, 0 \le p \le \infty$, $r, \mu \in N$, $\sum_{i=0}^{n-\mu-1} \omega_r\left(f \cup_{i=j}^{\mu} I_i\right)_p^p \le c\left(p, r, \mu\omega_r^{\varphi}\left(f, n^{-1}\right)_p^p\right)$.

Lemma 2.3. (Whitney's Inequality) (Burkill, 1952) Let $f \in L_p(I)$, $0 \le p \le \infty$. Then there exists $Q_n \in \Pi_n$, a polynomial of degree $\le n$ such that $\|f - Q_n\|_{L_p(I)} \le C\omega_r(f, |I|, I)_p$ for $r > n$.

Lemma 2.4. (Kopotun, 1997) For any polynomial $Q_n \in \Pi_n$, $0 \le p \le \infty$ we have $|J|^{1/p}\|Q_n\|_{L_p(I)} \sim \|Q_n\|_{L_p(I)}$.

Lemma 2.5. (Sendov and Popov, 1988) If f is bounded measurable function on $[a,b], a, b \in \Re$, then $\int_a^b f(x)dx \approx (b-a)n^{-1}\sum_{i=1}^{n} f(x_i)$ where $x_i = a + (b-a)(2i-1)/2n$.

The next lemma is proved in (Hu, 1995) by Hu for a continuous function, we can get a similar result for our case.

Lemma 2.6. For $f \in L_p[-1,1]$, $0 \le p \le \infty$, we have $E_n(f)_\infty \le \widetilde{E}_n(f)_\infty \le 2E_n(f)_\infty$.

Lemma 2.7. For $f \in L_p[-1,1]$, $0 \le p \le \infty$, we have $E_n(f)_p \le \widetilde{E}_n(f)_p \le 2E_n(f)_p$.

Proof

Denote by $\widetilde{P}$ and $\widetilde{Q}$ be the best onesided approximation, then by lemmas (2.4), (2.5) and (2.6), we have

$$
\begin{aligned}
&\widetilde{E}_n(f)_p^p = \left\|\widetilde{P} - \widetilde{Q}\right\|_p^p \\
&\le c\widetilde{E}_n(f)_\infty^p \\
&\le 2cE_n(f)_\infty^p \\
&= 2c\|f - P\|_\infty^p \\
&= 2\left(\sup_{x \in 1}|(f-P)(x)|\right)^p \\
&\le 2cn\sum_{i=1}^{n} n^{-1}|(f-P)(x_i)|^p \\
&\le 2cn\|f-P\|_p^p \\
&\le 2nE_n(f)_p^p
\end{aligned} \tag{9}
$$

where P is the best approximation of f. □

Also, it is clear that $\omega_r(f,t)_p \le \omega_r(f,t)_\infty$. But the next lemma shows that $\omega_r(f,t)_p$ and $\omega_r(f,t)_\infty$ are equivalent. Namely we prove the following result;

Lemma 2.8. For $0 \le p \le \infty$, $f \in L_p(I)$, we have $\omega_r(f,|I|)_\infty \le c(p)\omega_r^{\varphi}(f,|I|)_p$.

Proof

Denote $I_i, 0 \le i \le n-1$ a partition of the interval I. Let $Q_i \in \Pi_n$ be the best polynomial approximation of f on I_i, satisfying Whitney's inequality, such that

$$E_n(f)_{L_p(I_i)} = \|f - Q_i\|_{L_p(I_i)} \le c\omega_r(f, I_i)_{L_p(I_i)} \tag{10}$$

For $n \ge r$, then by definitions, lemmas (2.4), (2.2) and (2.7),

$$\begin{aligned}
&\omega_r(f,|I|)^p_{L_\infty(I)} = sub_{0<h\le|I|}\left\|\Delta_h^r(f,\cdot)\right\|^p_{L_\infty(I)} \\
&= \sum_{i=1}^n |I_i| \sup_{0<h\le|I|}\left\|\Delta_h^r(f,\cdot)\right\|^p_{L_\infty(I_i)} \\
&\le \sum_{i=1}^n |I_i| \sup_{0<h\le|I|}\left\|\Delta_h^r(f-Q_i,\cdot)\right\|^p_{L_\infty(I_i)} \\
&\le c\sum_{i=1}^n |I_i| \|f-Q_i\|^p_{L_\infty(I_i)} \\
&\le c\sum_{i=1}^n n^{-1}\widetilde{E}_i(f)^p_{L_\infty(I_i)} \\
&\le cn^{-1}\sum_{i=1}^n \widetilde{E}_i(f)^p_{L_{p(I_i)}} \\
&\le 2c\sum_{i=1}^n E_i(f)^p_{L_{p(I_i)}} \\
&= 2c\sum_{i=1}^n \|f-Q_i\|^p_{L_{p(I_i)}} \\
&\le 2c\sum_{i=1}^n \omega_r(f,I_i)^p_{L_{p(I_i)}} \\
&\le C\omega_r^\varphi(f,n^{-1})^p_{L_{p(I)}},
\end{aligned} \tag{11}$$

for $\widetilde{P}_i$ and $\widetilde{Q}_i$ be the best onesided approximation in $L_p(I_i)$, $0 < p \le \infty$, of degree less than n. □

The following Lemma proved by Devore, Leviatan and Yu (DeVore et. al., 1992) for the case $0 < p < 1$, and by Ditzian and Totik (Ditzian and Totik, 1978) for the other cases, that is, $1 \le p \le \infty$.

Lemma 2.9. For $E_n(f)_p$ with $0 < p \le \infty$, we have for all $n \ge r, E_n(f)_p \le c(r,p)\omega_r^\varphi(f,n^{-1})_p$.

The auxiliary lemma below allows us to blend local overlapping polynomials into a smooth spline with the same approximation order.

Lemma. 2.10. (Beatson Lemma) (Beatson, 1982) Let $n \ge 2$ be an integer and $d = 2(r-1)^2$. Let $T = \{t_i\}_{i=-\infty}^{\infty}$ be a strictly increasing knot sequence with $t_0 = a$, $t_d = b$. Let P, Q be two polynomials of degree less than r. Then there exists a spline $S \in \delta_r(T)$ such that

a. $S(x)$ is a number between $P(x)$ and $Q(x)$, for all $x \in [a,b]$.

b. $S = P$ on $(-\infty, a]$ and $S = Q$ on $[b, \infty)$.

The following lemma, which is proved in (Hu et. al., 1997) by Hu , Kopotun and Yu plays a main role in this chapter.

Lemma 2.11. Let $p \le \infty$, and let $S(x)$ be a spline of an odd order $r(r = 2m+1)$ on the knot sequence $\{x_j = \cos(j\pi/n)\}_{j\in J(Y_s)}$, where $n > c(Y_s)$ is such that there are at least four knots x_j in each interval (y_i, y_{i+1}), $i = 0,\cdots,s$, and $J(Y_s) = \{1,\cdots,n\} \setminus \{j, j-1 \mid x_j \le y_j < x_{j-1} \text{ for some } 1 \le i \le s\}$.

Then there exists an intertwining pair of polynomials $P_1, P_2 \in \Pi_{c(\mu)n}$ for S with respect to Y_s such that

$$\|P_1 - P_2\|_p^p \le C(r,\mu,s)^p \sum_{j=1}^{n-1} E_{r-1}(S, I_j \cup I_{j+1})_p^p, 0 < p < \infty \tag{12}$$

We need finally, for our proofs, the next property of Ditzian-Totik modulus of smoothness, which is proved in (Ditzian and Totik, 1978) by Ditzian and Totik, for $L_p[-1,1]$, $1 \le p \le \infty$. Also, Devore, Leviatan and Yu (DeVore et. al., 1992) showed that it is valid for $0 < p < 1$ as well.

Lemma 2.12. For $f \in L_p[-1,1]$, $0 \le p \le \infty$, we have $E_n(f)_{L_p[-1,1]} \le C\omega_r^{\varphi}(f, n^{-1})_{L_p[-1,1]}$.

3. Proof of the Main Theorems

3.1. Proof of Theorem I

The lower bound is clearly valid from definition.

For the upper bound, denote by $\widetilde{P}$ and $\widetilde{Q}$ best onesided approximation of f by polynomials of degree less than r from above and below in the space L_p. Then by definitions and lemmas (2.4), (2.12) and (2.8), we have

$$\begin{aligned}
\widetilde{E}_{r-1}(f)_p &= \|\widetilde{P}-\widetilde{Q}\|_p \le |I|^{\frac{1}{p}} \|\widetilde{P}-\widetilde{Q}\|_\infty \\
&= |I|^{\frac{1}{p}} \widetilde{E}_{r-1}(f)_\infty \\
&\le 2|I|^{\frac{1}{p}} E_{r-1}(f)_\infty \\
&\le 2|I|^{\frac{1}{p}} \omega_r^{\varphi}(f, |I|)_\infty \\
&\le C\omega_r^{\varphi}(f, |I|) p. \qquad (13)
\end{aligned}$$

3.2. Proof of Theorem II

Let $d := 2(r-1)^2$, $m := [(n+d-1)/d]$ and $\bar{z}_i := z_{di}$. Note that $\bar{z}_i := -1$ for $i \le 0$ and $\bar{z}_i := 1$ for $i \ge m$. We first construct overlap-ing polynomial pieces of degree less than r on the coarser partition $\overline{T}_n := \{\bar{z}_i\}_{i=0}^m$.

We call the interval $\bar{I}_i[\bar{z}_i, \bar{z}_{i+1}]$ contaminated if $\bar{z}_i < y_i < \bar{z}_{i+1}$ for some $y_i \in Y_s$. By assumption, there exists exactly one y_i in each of the contaminated interval $\bar{I}_{m_j}$, $j = 1, \cdots, s$ and there is at least one non-contaminated interval between $\bar{I}_{m_j}$ and $\bar{I}_{m_{j+1}}$, that is $m_j < m_j + 2 \le m_{j+1}$, $j = 1, \cdots, s-1$.

If $m_{j+1} = m_{j+2}$ (i.e., if there is only one non-contaminated interval between $\bar{I}_{m_j}$ and $\bar{I}_{m_{j+1}}$), then the following construction is not needed, and the next two paragraphs can be skipped .

In this case $m_j + 2 \le m_{j+1}$, by Whitney's Theorem for Onesided Approximation (Theorem I) on each of the interval $[\bar{z}_i, \bar{z}_{i+2}]$, $i = m_j + 1, \cdots, m_{j+1} - 2$, there exist two polynomials P_i and Q_i of degree less than r such that $P_i(x) \ge f(x) \ge Q_i(x)$ for all $x \in [\bar{z}_i, \bar{z}_{i+2}]$ and

$$\|P_i - Q_i\|_{L_p[\bar{z}_i, \bar{z}_{i+2}]} \le \omega_r^{\varphi}(f, |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_p. \qquad (14)$$

We define p_i and q_i on $[\bar{z}_i, \bar{z}_{i+2}]$ by $p_i := P_i$ and $q_i := Q_i$ if $(-1)^{s-j} > 0$, and $p_i := Q_i$ and $q_i := P_i$ if $(-1)^{s-j} < 0$.

Hence $(-1)^{s-j}(p_i(x) - f(x)) \ge 0$, $(-1)^{s-j}(q_i(x) - f(x)) \le 0$ and

$$\begin{aligned}
\|p_i - q_i\|_{L_p[\bar{z}_i, \bar{z}_{i+2}]} &= \|P_i - Q_i\|_{L_p[\bar{z}_i, \bar{z}_{i+2}]} \\
&\le c\omega_r^{\varphi}(f, |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_p \\
&\le c\omega_{r-1}^{\varphi}(f', |\bar{I}_i|, [\bar{z}_i, \bar{z}_{i+2}])_{p'} \qquad (15)
\end{aligned}$$

where, in this step we have used lemma (2.1).

We should emphasize that when we speak of a polynomial on an interval we mean the restriction to the interval; hence it is considered undefined outside. Near each point y_i, we construct local polynomials differently. More precisely, we approximate f' on $\left[\bar{z}_{m_{j-1}}, \bar{z}_{m_{j+2}}\right]$, $j = 1, \cdots, s$, from above and below by two polynomials $\tilde{P}_{m_j}$ and $\tilde{Q}_{m_j}$ of degree less than $r-1$. Then $\tilde{P}_{m_j}(x) \ge f'(x) \ge \tilde{Q}_{m_j}(x)$ for all $x \in \left[\bar{z}_{m_{j-1}}, \bar{z}_{m_{j+2}}\right]$ and

$$\left\|\tilde{p}_{m_j} - \tilde{q}_{m_j}\right\|_{L_p\left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right]} \le c\omega_{r-1}^{\varphi}(f', \left|\bar{I}_{m_j}\right|, \left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right])_p. \tag{16}$$

Define $\tilde{p}_{m_j} := \tilde{P}_{m_j}$ and $\tilde{q}_{m_j} := \tilde{Q}_{m_j}$ if $(-1)^{s-j} > 0$, and $\tilde{p}_{m_j} := \tilde{Q}_{m_j}$ and $\tilde{q}_{m_j} := \tilde{P}_{m_j}$ otherwise.

It's easy to check that $p_{m_j} = \int_{y_i}^{x}\int_{y_i}^{t_2} \tilde{P}_{m_j}(t_1)dt_1dt_2 + f(y_i)$ and $q_{m_j} = \int_{y_i}^{x}\int_{y_i}^{t_2} \tilde{Q}_{m_j}(t_1)dt_1dt_2 + f(y_i)$ satisfy the inequalities

$$\left\|p_{m_j} - q_{m_j}\right\|_p = \left\|\int_{y_i}^{x}\int_{y_i}^{t_2}\left[\tilde{P}_{m_j}(t_1) - \tilde{Q}_{m_j}(t_1)\right]dt_1dt_2\right\|_{L_p\left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right]}$$

$$\le \left\|\int_{\bar{z}_{m_j-1}}^{\bar{z}_{m_j+2}}\int_{\bar{z}_{m_j-1}}^{t_2}\left[\tilde{P}_{m_j}(t_1) - \tilde{Q}_{m_j}(t_1)\right]dt_1dt_2\right\|_{L_p\left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right]}$$

$$\le \left(\int_I\left(\int_{\bar{z}_{m_j-1}}^{\bar{z}_{m_j+2}}\int_{\bar{z}_{m_j-1}}^{t_2}\left[\tilde{P}_{m_j}(t_1) - \tilde{Q}_{m_j}(t_1)\right]dt_1dt_2\right)^p dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_I\left(\int_{\bar{z}_{m_j-1}}^{\bar{z}_{m_j+2}} c(m_j)\left|\bar{I}_{m_j}\right|\left(\tilde{P}_{m_j}(t_1) - \tilde{Q}_{m_j}(t_1)\right)dt_1dt_2\right)^p dx\right)^{\frac{1}{p}}$$

$$\le \left(\int_I\left(c(m_j)\left|\bar{I}_{m_j}\right|^2\left\|\tilde{P}_{m_j} - \tilde{Q}_{m_j}\right\|_{L_\infty\left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right]}\right)^p dx\right)^{\frac{1}{p}}$$

$$\le c(m_j)\left|\bar{I}_{m_j}\right|^2\left\|\tilde{P}_{m_j} - \tilde{Q}_{m_j}\right\|_{L_p\left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right]}$$

$$\le c(m_j)\left|\bar{I}_{m_j}\right|^2 \omega_{r-1}^{\varphi}(f', \left|\bar{I}_{m_j}\right|, \left[\bar{z}_{m_j-1}, \bar{z}_{m_j+2}\right])_p. \tag{17}$$

Having constructed the overlapping local polynomials which are "intertwining" with f and have the right approximation order, we now blend them for smooth spline approximation $\bar{S}$ and S on the original knot sequence T_n with the same properties. If both $\bar{I}_{i-1}$ and $\bar{I}_i$ are non contaminated and $i < m$, then p_{i-1} and p_i overlap on $\bar{I}_i$, which contains $d-1$ interior knot from T_n.

By Beatson's lemma (2.10), there exists a spline $\bar{S}_i$ of order r on $\bar{I}_i$ on these knots connects with p_{i-1} and p_i in a C^{r-2} manner at $\bar{z}_i = z_{di}$ and $\bar{z}_{i+1} = z_{d(i+1)}$, respectively.

Moreover, the graph of $\bar{S}_i$ lies between those of p_{i-1} and p_i, and hence $\operatorname{sgn}(p_{i-1}(x) - f(x)) = \operatorname{sgn}(p_i(x) - f(x)) = \operatorname{sgn}(\bar{S}_i(x) - f(x))$, $x \in \bar{I}_i$.

Similarly, considering the overlapping polynomials q_{i-1} and q_i, we construct a spline S_i satisfying $\operatorname{sgn}(q_{i-1}(x) - f(x)) = \operatorname{sgn}(q_i(x) - f(x)) = \operatorname{sgn}(S_i(x) - f(x))$, $x \in \bar{I}_i$.

Also,

$$\int_{\bar I_i}\left|\bar S_i - S_i\right|^p \le 2^p\left(\int_{\bar I_i}\left|p_{i-1}-q_{i-1}\right|^p + \int_{\bar I_i}\left|p_i - q_i\right|^p\right). \tag{18}$$

By (15), this gives

$$\left\|\bar S_i - S_i\right\|_{L_p(\bar I_i)} \le c\omega_r^{\varphi}(f, |\bar I_i|, [\bar z_{i-1}, \bar z_{i+2}])_p$$

$$\le c|\bar I_i|^2 \omega_r^{\varphi}(f', |I_i|, [\bar z_{i-1}, \bar z_{i+2}])_p. \tag{19}$$

The blending of the overlapping polynomial pieces involving contaminated intervals can be done in the same way. The spline pieces $\bar S_i$ and S_i thus produced also satisfy the estimate above with a slightly larger interval in place of $[\bar z_{i-1}, \bar z_{i+2}]$ on the right-hand side ($[\bar z_{i-2}, \bar z_{i+3}]$ at worst), which will make no difference in the rest of the proof. We define the final spline $\bar S$ on each $\bar I_i$ as follows;

If there is only one local polynomial p_i over $\bar I_i$, set $\bar S$ to this polynomial, if there are two polynomials overlapping on $\bar I_i$, then there must be a blending local Spline $\bar S_i$, set $\bar S$ to $\bar S_i$. It is clear from its construction that $\bar S - f \in \Delta^0(Y_s)$ on the whole interval $[-1,1]$, and $\bar S \in C^{r-2}$. Similarly, we construct $S \in C^{r-2}$ such that $f - S \in \Delta^0(Y_s)$.

Now, recall that all neighboring intervals $I_i = [z_i, z_{i+1}]$ in the original partition T_n are comparable in size and each interval $\bar I_i = [z_{di}, z_{d(i+1)}]$ contains no more than d such intervals. Therefore, the inequality (1) follows directly from (14) and (15).

Now, (2) is a direct consequence of the previous inequality and (2.1). ■

3.3. Proof of Theorem IV

It follows from Corollary III (with $T_n = \{x_j\}$) that there exist splines $\bar S$ and S of an integer r such that $\bar S(x) \ge f(x) \ge S(x)$, $x \in I$ and

$$\left\|\bar S - S\right\|_{L_p(\bar I_i)} \le c\omega_r^{\varphi}(f, |I_j|, I_j)_p. \tag{20}$$

Since $|\mathrm{I}_j| \sim |I_j| = h_j = \Delta(x)$ and $x \in I$, then

$$E_{r-1}(\bar S, I_j \cup I_{j+1})_p \le E_{r-1}(\bar S - f, I_j \cup I_{j+1})_p + E_{r-1}(f, I_j \cup I_{j+1})_p$$

$$\le \left\|\bar S - S\right\|_{L_p(I_j \cup I_{j+1})} + c\omega_r^{\varphi}(f, h_j)_p$$

$$\le c\omega_r^{\varphi}(f, h_j)_p. \tag{21}$$

Hence

$$E_{r-1}(\bar S, I_j \cup I_{j+1})_p \le c\omega_r^{\varphi}(f, h_j)_{p'} \tag{22}$$

And similarly,

$$E_{r-1}(S_0, I_j \cup I_{j+1})_p \le c\omega_r^{\varphi}(f, h_j)_p. \tag{23}$$

Lemma (2.11) implies the existence of the polynomials $\bar P_1, \bar P_2, P_1$ and P_2 of degree less than or equal to $c(m)n$ such that $\bar P_1(x) \ge \bar S(x) \ge \bar P_2(x)$, $P_1(x) \ge S(x) \ge P_2(x)$ and by using (2.2) we get

$$\left\|\bar P_1 - \bar P_2\right\|_p^p \le c^p \sum_{j=1}^{n-1} E_{r-1}(S, I_j \cup I_{j+1})_p$$

$$\le c^p \sum_{j=1}^{n-1} \omega_r^{\varphi}(f, h_j)_p^p$$

$$\le c(r, \mu, s, p, m)\omega_r^{\varphi}(f, n^{-1})_{p'}^p \tag{24}$$

And similarly,

$$\left\|P_1 - P_2\right\|_p^p \le \omega_r^{\varphi}(f, n^{-1})_p^p \tag{25}$$

Now, the polynomials $\overline{P}_1$ and P_2 are what we are looking for, since $\overline{P}_1 \geq \overline{S} \geq f \geq S \geq P_2$ and

$$\left\|\overline{P}_1 - P_2\right\|_p^p \leq \left\|\overline{P}_1 - \overline{P}_2\right\|_p^p + \left\|\overline{S} - S\right\|_p^p + \left\|P_1 - P_2\right\|_p^p$$

$$\leq C\omega_r^{\varphi}(f, n^{-1})_p^p. \tag{26}$$

Inequality (5) holds and the proof is completed. ■

3.5. Proof of Theorem V

Theorem II implies the existence of intertwining pair of spline $\{\overline{S}, S\}$ of order r for f on the knot sequence $\{x_j\}_{j \in J(Y_s)}$, recall that $J(Y_s) = \frac{\{1, \cdots, n\}}{\{j, j-1 \mid x_j \leq y_j \leq x_{j-1} \text{for some } 1 \leq i \leq s\}}$ satisfying

$$\left\|\overline{S} - S\right\|_{L_p(I_j)} \leq c\left|I_j\right|\omega_{r-1}^{\varphi}(f', \left|I_j\right|, I_j)_{p'} \tag{27}$$

Where r is an odd integer such that $m+1 \leq r \leq m+2$.

Now, Since $\Delta_n(x) \sim n^{-1}$, therefore

$$\left\|\overline{S} - S\right\|_p \leq cn^{-1}\omega_{r-1}^{\varphi}(f', n^{-1})_p \tag{27}$$

Now, theorem IV implies that there exist intertwining pairs of polynomials $\{\overline{P}_1, \overline{P}_2\}$ and $\{P_1, P_2\}$ for $\overline{S}$ and S, respectively, satisfying the inequalities (24) and (25) as the previous proof.

Finally, $\{\overline{P}_1, P_2\}$ is an intertwining pair of polynomials for f satisfying the inequalities (6), (7) and (8).

4. Conclusion

We have improved some results on onesided and co-onesided polynomial and spline approximation. Also, we have relaxed some shape constrains in small parts of the interval $[-1,1]$, and approximate a function f in the space $L_p^r[-1,1]$, $0 < p \leq \infty$, r =0, 1, or 2 by co-onesided pair of splines and/or polynomials to get global estimates with less degree of approximation in terms of Ditzian-Totik modulus of smoothness.

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