t-Extending Semimodule over Semiring

Samah Alhashemi
College of Science for Women,
University of Babylon,
Babylon, Iraq
Wsci.samah.abid@uobabylon.edu.iq
https://orcid.org/0000-0002-2411-3953

Asaad M. A. Alhossaini
College of Education for Pure Sciences,
Babylon University,
Babylon, Iraq
asaad_hosain@itnet.uobabylon.edu.iq
https//orcid.org/0000-0002-4569-3352

***Abstract*—The main aim of this research is to present and study several basic characteristics of the idea of t-extending semimodules. The semimodule *F* is said to be a t-extending semimodule if each t-closed sub-semimodule of *F* is t-essential in a direct summand of *F*. Hence, the behavior of the t-extending semimodule is considered. In addition, the relationship between the t-essential (t-closed) and essential (closed) has been studied and investigated as well. Finally, in this work, there are a number of results related to the t-extending property, which is one of the generalizations of extending property, (every extending is t-extending, while the converse is not true).**

Keywords*— t-essential subsemimodule, t-closed subsemimodule, t-extending semimodule, extending semimodule, z2-torsion semimodule.

I. INTRODUCTION

In this work, the t-extending semimodule over a semiring will be introduced and investigated. Throughout this paper, R will denote a commutative semiring with identity, and F is a left R-semimodule. A **semiring** is a non-empty set R with two operations of addition (+) and multiplication (·) such that (R, +) is a commutative monoid with identity element 0; (R, ·) is a monoid with identity element $1 \neq 0$; $r0 = 0r = 0$ for all $r \in R$; $a(b + c) = ab+ac$ and $(b + c)a = ba + ca$ for every $a, b, c \in R$. We say that R is a commutative semiring if the monoid (R, ·) is commutative [1]. Let (F, +) be an additive abelian monoid with additive identity 0_F, then F is called a left ***R*-semimodule** if there exists a scalar multiplication $R \times F \longrightarrow F$ denoted by $(r, f) \mapsto rf$, such that $(rr')f = r(r'f)$; $r(f + f') = rf + rf'$; $(r + r')f = rf + r'f$; $1f = f$ and $r0_F = 0_R f = 0_F$ for all $r, r' \in R$ and all $f, f' \in F$[2].

A subset A of an R-semimodule F is called a **subsemimodule** of F if for $a, a' \in A$ and $r \in R$, $a + a' \in A$ and $ra \in A$ and write ($A \leq F$)[3]. A nonzero R-subsemimodule A of F is said to be **essential** (large) and write ($A \leq^e F$)if $A \cap L \neq 0$ for every nonzero subsemimodule L of F [4]. A subsemimodule A of a semimodule F is said to be **closed** if $A \leq^e A' \leq F$ implies $A = A'$ (denoted by $A \leq^c F$) [5].

A subsemimodule Z (F) of F is defined by $Z(F)=\{f \in F | \text{ann}(f) \leq^e R\}$ is said to be singular subsemimodule of F. If $Z(F) = F$, then F is called singular. If $Z(F) = 0$, then F is called nonsingular [6]. The second singular subsemimodule $Z_2(F)$ of F is that subsemimodule of F, containing $Z(F)$, such that $Z_2(F)/Z(F)$ is the singular subsemimodule of $F/Z(F)$ [7]. A subsemimodule A of F is said to be t-essential and write ($A \leq^{tes} F$) if for any $C \leq F$, $A \cap C \leq Z_2(F)$, implies $C \leq Z_2(F)$. A subsemimodule C of F is said to be t-closed and write ($C \leq^{tc} F$) if $C \leq^{tes} C' \leq F$ implies $C = C'$. An R- semimodule F is said to be t-extending if every t-closed subsemimodule of F is a direct summand of F.

The paper is further organized as follows: Section 2 studies the t-essential and t-closed subsemimodule. In section 3, the t-extending semimodule is introduced and studied, proving some of its properties.

II. T-ESSENTIAL AND T-CLOSED SUBSEMIMODULE

The properties of t-essential and t-closed subsemimodules are introducing and investigating in this section. Through the analysis of the structure of the t-essential and t-closed subsemimodules, it can be observed that there are many properties of nonsingular and Z_2-torsion semimodule that are also useful. In the next, there are some properties of nonsingular and Z_2-torsion submodule, those properties will be converted for the subsemimodule.

Definition 1: A subsemimodule A of F is said to be **t-essential** and write ($A \leq^{tes} F$) if for any $C \leq F$, $A \cap C \leq Z_2(F)$, implies $C \leq Z_2(F)$**.**

Definition 2: A subsemimodule C of F is said to be **t-closed** and write ($C \leq^{tc} F$) if $C \leq^{tes} C' \leq F$ implies $C=C'$.

Lemma 3: If F is nonsingular then $Z_2(F) = 0$.

Proof: Assume F is nonsingular, then $Z(F) = 0$, since $Z_2(F)/Z(F) = Z(F/Z(F))=0$, then $Z_2(F) = Z(F) = 0$.

Note that: If F is singular then $Z_2(F) = Z(F) = F$.

Remark 4: For any semimodule F:

1. Every essential subsemimodule of F is *t*- essential**.**
2. If $A \cap Z_2(F) = 0$, then A is t- essential.
3. Every t-closed subsemimodule of F is closed.
4. If $A \leq^c F$, and F is nonsingular then $A \leq^{tc} F$.
5. For each $A \leq F$, $0 \leq^{tes} Z_2(A)$, in particular $0 \leq^{tes} Z_2(F)$.
6. F is nonsingular if and only if $0 \leq^{tc} F$.
7. $Z_2(F) \leq^{tc} F$**.**
8. $F/Z_2(F)$ is nonsingular.

Proof: **for (1)**, Clear**.**

Note that: The converse of (1) is not true. For example, in Z_6, ⟨2⟩ and ⟨3⟩ are t-essential subsemimodules but not essential.

(2), Let $A \cap Z_2(F) = 0$. Hence for each B with B is not subsemimodule of$Z_2(F)$, $Z_2(F) \cap (A \cap B) = 0$, then $A \cap B$ is not subsemimodule of $Z_2(F)$(if not $Z_2(F) \cap (A \cap B) = A \cap B \neq 0$.

(3), assume $A \leq^{tc} F$ and let $A \leq^{e} B \leq F$, then by (1),$A \leq^{tes} B$, but $A \leq^{tc} F$, then A=B, and hence $A \leq^{c} F$.

Note that: The converse of (3) is not true for example in Z_6, ⟨2⟩ and ⟨3⟩are closed subsemimodules but not t-closed.

(4), assume that $A \leq^{c} F$, and $A \leq^{tes} B \leq F$, then for each $C \leq B$, $A \cap C \leq Z_2(F) = 0$ (since F is nonsingular and by Lemma 3) implies C=0, that is, $A \leq^{e} B$, but $A \leq^{c} F$, by assumption , so A=B, and $A \leq^{tc} F$.

(5), Clear.

(6), since $0 \leq^{c} F$ (always true), by (4), $0 \leq^{tc} F$. Conversely, assume that $0 \leq^{tc} F$, and F is not nonsingular then by Lemma.3 $Z_2(F) \neq 0$, and by (5) $0 \leq^{tes} Z_2(F)$, a contradiction, hence F is nonsingular.

(7), assume $Z_2(F) \leq^{tes} F' \leq F$, let $B \leq F'$and$B \nsubseteq Z_2(F)$, then $Z_2(F) \cap B \leq Z_2(F)$, contradiction, then $Z_2(F) = F'$.

(8), since $0 = (Z_2(F)/ Z_2(F)) \leq^{tc} (F/ Z_2(F))$, then by (6), $F/ Z_2(F)$ is nonsingular.

***Corollary* 5**: For any semimodule F, $Z_2\big(F/Z_2(F)\big) = 0$.

Proof: Clear by Remark 4(8) and Lemma 3 **.**

Definition 6: A semimodule $F$is said to be Z_2 –torsion if $Z_2(F) = F$.

Lemma 7: Every singular semimodule F is Z_2–torsion.

Proof: Let F be singular semimodule, since $Z(F) \leq Z_2(F) \leq F$, and $Z(F) = F$, then $Z_2(F) = F$. □

Lemma 8: A subsemimodule A of F is Z_2-torsion if and only if $A \leq Z_2(F)$.

Proof: Suppose that A is Z_2-torsion, then $Z_2(A) = A$, but $Z_2(A) \leq Z_2(F)$, then $A \leq Z_2(F)$.

Conversely, assume that $A \leq Z_2(F)$, since by [8] $Z_2(A) = A \cap Z_2(F) = A$, therefore A is Z_2-torsion .

Lemma 9: Let A be subsemimodule of F. If A and F/A are Z_2-torsion, then F is Z_2-torsion.

Proof: Let $x \in F$, then $x + A \in F/A$, by assumption $x + A \in Z_2(F/A)$, then $x + A + Z(F/A) \in Z(F/A/Z(F/A))$, so there exists $I \leq^{e} R$, such that $I(x + A + Z(F/A))=0$, therefore $I(x + A) \leq Z(F/A)$, hence there exists $J \leq^{e} R$ such that $(I \cap J)(x + A)=0$, then $(I \cap J)(x) \leq A = Z_2(A)$ [by assumption], since $Z_2(A) \leq Z_2(F)$ then $(I \cap J)\, x \leq Z_2(F)$, this implies, $x + Z_2(F) \in Z_2(F/Z_2(F))$, but by Corollary 5, $Z_2(F/Z_2(F) = 0$, hence $x \in Z_2(F)$, but $Z_2(F) \leq F$, then $Z_2(F) = F$ and F is Z_2-torsion.

Proposition 10: If $A \leq F$, and F is nonsingular then A is t-essential if and only if A is essential in F.

Proof: Assume that A is t-essential in F, and let $A \cap B$=0, where $B \leq F$. Since A is t-essential in F, then $B \leq Z_2(F)$, since F is nonsingular then by Lemma 3, $Z_2(F) = 0$ therefore B=0, and so A is an essential in F. Conversely, clear by Remark 4(1).

Note that: If F is singular then any subsemimodule of F is t-essential.

Proposition 11: For a subsemimodule A of F. If $(A + Z_2(F)) \leq^{e} F$ then F/A is Z_2-torsion.

Proof: Assume that $(A + Z_2(F)) \leq^{e} F$, then by [8], $F/(A + Z_2(F))$ is singular, and hence by Lemma 7, $F/(A + Z_2(F))$ is Z_2-torsion. But $(A + Z_2(F)\ )/\ A \cong Z_2(F)/(A \cap Z_2(F)) = Z_2(F)/Z_2(A)$ is singular, hence by Lemma 7, $(A + Z_2(F)\ /\ A$ is Z_2-torsion, and $(F/A)/\ ((A + Z_2(F)/ A) \cong F/(A + Z_2(F))$ is Z_2-torsion, then by Lemma 9, F/A is Z_2-torsion.

Proposition 12: If F/A is Z_2-torsion, then A is t-essential in F.

Proof: Assume that F/A is Z_2-torsion, since $(F/A)/Z(F/A) = (Z_2(F/A))/(Z(F/A)) = Z((F/A)/Z(F/A))$, then $(F/A)/Z(F/A)$is singular, but $(F/A)/Z(F/A) \cong F/A^*$, where $A^*/A = Z(F/A)$, so F/A^* is singular. Now let $A \cap B \leq Z_2(F)$, and $b \in B \leq F$, then $b \in F$, so $b + A^* \in F/A^* = Z(F/A^*)$, then there exists $I \leq^{e} R$, such that $I(b + A^*)=0$. Therefore $Ib \leq A^*$, for every $x \in I$, $xb + A \in A^*/A$, since $A^*/A = Z(F/A)$, then there exists $K \leq^{e} R$, such that $K(xb + A)=0$, so $Kxb \leq A$, but $Kxb \leq B$, so $Kxb \leq A \cap B \leq Z_2(F)$, thus $xb + Z_2(F) \in Z(F/Z_2(F))$ =0 hence $Ib \leq Z_2(F)$, so $b + Z_2(F) \in Z(F/Z_2(F))$ =0, so $b \in Z_2(F)$, and hence $B \leq Z_2(F)$, so A is t-essential in F.

A subsemimodule B of a semimodule F is called *complement* of a subsemimodule A in F if $B \cap A = 0$ and B is a maximal with this property [5].

Proposition 13: If $A \leq F$, then the following statements are equivalent:

1. A is t-essential in F.
2. $A + Z_2(F) \leq^{e} F$.
3. $(A + Z_2(F))/Z_2(F) \leq^{e} F/Z_2(F)$.

Proof: $(1 \Longrightarrow 2)$, Assume that A is t-essential in F, and B is a complement of A in F so $A + B \leq^{e} F$. Since A is t-essential in F, then $B \leq Z_2(F)$, but $A + B \leq A + Z_2(F)$, and $A + B \leq^{e} F$, therefore $A + Z_2(F) \leq^{e} F$.

$(2 \Longrightarrow 3)$, Assume that $A + Z_2(F) \leq^{e} F$ since by [8] $Z_2(F) \leq^{c} F$, then by [9] $(A + Z_2(F))/Z_2(F) \leq^{e} F/Z_2(F)$.

$(3 \Longrightarrow 1)$ Clear by Propositions 11 and 12.

Lemma 14: Let F and F' be semimodules and $f: F \to F'$, be an epimorphism. If F is Z_2- torsion then F' is Z_2 torsion.

Proof: Assume that $f: F \to F'$, is an epimorphism and F is Z_2 torsion, since $F' = f(F) = f(Z_2(F)) \subseteq Z_2(F')$, therefore F' is Z_2 torsion.

Note that: If F is Z_2- torsion, then $F/Z(F) = Z_2(F)/Z(F) = Z(F/Z(F)\ )$, therefore $F/Z(F)$ is singular hence by Lemma 7, $F/Z(F)$is Z_2- torsion.

Proposition 15: Let F be an R-semimodule. If $A \leq^{tc} F$, then $Z_2(F) \leq A$.

Proof: Assume that $A \leq^{tc} F$, since $(A + Z_2(F))/A \cong Z_2(F)/(A \cap Z_2(F))$, by Lemma 14,

$(A+Z_2(F))/A$ is Z_2- torsion, hence by proposition 12, $A \leq^{tes} (A+Z_2(F))$, but $A \leq^{tc} F$, by assumption, then $A = A+Z_2(F)$, hence $Z_2(F) \leq A$.

Proposition 16: For a semimodule F. If $C \leq A$, then $A \leq^{tc} F$ if and only if $A/C \leq^{tc} F/C$.

Proof: Assume that $A \leq^{tc} F$, and A/C is not t-closed in F/C, then there exists $F'/C \leq F/C$ such that $A/C \leq^{tes} F'/C$, then by Propositions 13 and 11, (F'/A) is Z_2 torsion, hence by Proposition 12 $A \leq^{tes} F'$, a contradiction with the assumption that $A \leq^{tc} F$. Conversely, suppose that $A/C \leq^{tc} F/C$, and A is not t-closed in F, then there exists $F' \leq F$, such that $A \leq^{tes} F'$, then by Proposition 13, $(A+Z_2(F')) \leq^{e} F'$ hence by Proposition 11 F'/A is Z_2 torsion, then $(F'/C)/(A/C)$ is Z_2 torsion. By Proposition 12, $A/C \leq^{tes} F'/C$, a contradiction, therefore $A \leq^{tc} F$.

Proposition 17: Let F be an R-semimodule and $A \leq F$. If there exists subsemimodule S such that A maximal with respect to property that $S \cap A$ is a Z_2 torsion, then $A \leq^{tc} F$.

Proof: Suppose the property of A hold, and let $A \leq^{tes} F' \leq F$, then $A \cap (F' \cap S) \leq Z_2(F)$, implies $(F' \cap S) \leq Z_2(F)$, therefore $F' \cap S$ is Z_2 torsion, but A is maximal with this property then $A = F'$, and hence $A \leq^{tc} F$.

Proposition 18: Let F be an R-semimodule and $A \leq F$. If $A \leq^{tc} F$, then A contains $Z_2(F)$,and $A/Z_2(F) \leq^{c} F/Z_2(F)$.

Proof: Suppose that $A \leq^{tc} F$, then by Proposition 15, $Z_2(F) \leq A$, now let $(A/Z_2(F)) \leq^{e} (F'/Z_2(F)) \leq (F/Z_2(F))$, then by[9], $A \leq^{e} F'$, but by Remark 4 (3), $A \leq^{c} F$, a contradiction, hence $A/Z_2(F) \leq^{c} F/Z_2(F)$.

Proposition 19: Let F be an R-semimodule and $A \leq F$. If $Z_2(F) \leq A$, and $A/Z_2(F) \leq^{c} F/Z_2(F)$, then $A \leq^{c} F$.

Proof: Suppose that $A/Z_2(F) \leq^{c} F/Z_2(F)$, with $Z_2(F) \leq A$, and let $A \leq^{e} F' \leq F$, then by Proposition 13, $A/Z_2(F) \leq^{e} F'/Z_2(F)$ a contradiction, then $A=F'$ and $A \leq^{c} F$.

Proposition 20: Let F be an R-semimodule. If $Z_2(F) \leq A$, and $A \leq^{c} F$, then A is a complement of nonsingular subsemimodule of F.

Proof: Let A be a complement of F' in F, hence by[8] $Z_2(F') = F' \cap Z_2(F) = 0$, then F' is nonsingular.

Proposition 21`: Let F be an R-semimodule and $A \leq F$, then $A \leq^{tc} F$ if and only if F/A is nonsingular.

Proof: It is clear by Proposition 16 and Remark 4 (6).

Recall that, a homomorphism R-semimodule $\varphi : A \longrightarrow B$ is said to be ***k- regular*** if $\varphi(a) = \varphi(a')$ then $a + k = a' + k'$ for some a, a'$\in A$ and k, $k' \in \ker(\varphi)$[10]. A ***subtractive*** subsemimodule K is a subsemimodule of F such that if k, $k+t \in K$ then $t \in K$ [11]. A semimodule F is **additively cancellative** if for all $a, a'$and $a'' \in F$, with $a+a' = a+a''$ implies $a' = a''$ [10]. A semimodule F is said to be **semi subtractive**, if for any f, $f' \in F$ there is always some $h \in F$ Satisfying $f+h = f'$ or $f'+h = f$ [2].

Corollary 22: For any semisubtractive and cancellative R-semimodule F. If $\emptyset$ is a k- regular endomorphism of F and A is a t-closed subtractive subsemimodule of F, then $\emptyset^{-1}(A) \leq^{tc} F$.

Proof: Let $\theta: F/\emptyset^{-1}(A) \to F/A$ such that $\theta: m + \emptyset^{-1}(A) \to \emptyset(m) + A$ [θ is well defined since $m_1 + \emptyset^{-1}(A) = m_2 + \emptyset^{-1}(A)$, then $m_1 + h_1 = m_2 + h_2$, for some $h_1, h_1 \in \emptyset^{-1}(A)$, then $\emptyset(m_1) + \emptyset(h_1) = \emptyset(m_2) + \emptyset(h_2)$, where $\emptyset(h_1), \emptyset(h_2) \in A$,hence $\emptyset(m_1) + A = \emptyset(m_2) + A$. On other hand if $\emptyset(m_1) + A = \emptyset(m_2) + A$, then $\emptyset(m_1) + a_1 = \emptyset(m_2) + a_2$, where $a_1, a_2 \in A$. By semi subtractive there exists t such that either $m_1+t=m_2$ or $m_1=m_2+t$. Case one: $m_1+t=m_2$, by cancellative,$a_1 = \emptyset(t) + a_2$, so by subtractive $\emptyset(t) \in A$. Case two: $m_1=m_2+t$, by cancellative, $\emptyset(t) + a_1 = a_2$ by subtractive $\emptyset(t) \in A$, hence $t \in \emptyset^{-1}(A)$. Therefore $\emptyset(m_1) + \emptyset(t) = \emptyset(m_2) + \emptyset(t)$, so $\emptyset(m_1 + t) = \emptyset(m_2 + t)$. Since $\emptyset$ is k- regular, then $m_1 + t + k = m_2 + t + k'$), where k, $k' \in \ker \emptyset$, since $\ker\emptyset \leq \emptyset^{-1}(A)$, hence $m_1 + \emptyset^{-1}(A) = m_2 + \emptyset^{-1}(A)$, therefore θ is monomorphism hence $F/\emptyset^{-1}(A) \cong$ subsemimodule of F/A (which is nonsingular), then $F/\emptyset^{-1}(A)$ is nonsingular hence by Proposition 21, $\emptyset^{-1}(A) \leq^{tc} F$.

Note that: If A is closed subsemimodule of F, then $\emptyset^{-1}(A) \leq^{c} F$.

Corollary 23: Let F be an R-semimodule. If $A \leq^{tc} F$, then $A = Z_2(F)$, if and only if A is Z_2 torsion.

Proof: Assume that $A \leq^{tc} F$, then by Proposition 15, $Z_2(F) \leq A$, but A is Z_2 torsion implies $A = Z_2(A) \leq Z_2(F)$, then $A = Z_2(F)$. Conversely, $A = Z_2(F)$ implies $Z_2(A) = A \cap Z_2(F) = A$, that is, A is Z_2 torsion.

Corollary 24: Let F be an R-semimodule and $A \leq^{tc} F$, then A is Z_2 torsion if and only if there exists a t-essential subsemimodule S of F for which $A \cap S \leq Z_2(F)$.

Proof: Assume that $A \leq^{tc} F$ and A is Z_2 torsion, then by Corollary 23, $A = Z_2(F)$ and $A \cap F \leq Z_2(F)$, where F is t-essential subsemimodule of F.

Conversely, assume that $S \leq^{tes} F$, and $A \cap S \leq Z_2(F)$, then $A \leq Z_2(F)$, but by Proposition 15, $Z_2(F) \leq A$, hence $A = Z_2(F)$, since by [8] $Z_2(A) = A \cap Z_2(F) = A$, then A is Z_2 torsion.

Proposition 25: Let F be an R-semimodule and $A \leq N \leq F$. If $A \leq^{tc} F$, then $A \leq^{tc} N$.

Proof: Assume that $A \leq^{tc} F$, and $A \leq^{tes} N' \leq N$, then for each $N'' \leq N'$, $A \cap N'' \leq Z_2(N')$, implies $N'' \leq Z_2(N)$, since $Z_2(N') \leq Z_2(N)$ hence $A \leq^{tes} N$, a contradiction , then $A \leq^{tc} N$.

Proposition 26: Let F be an R-semimodule and $A \leq N \leq F$. If $A \leq^{tc} N$, and $N \leq^{tc} F$ then $A \leq^{tc} F$.

Proof: Assume that $A \leq^{tc} N$, and $N \leq^{tc} F$, then by Proposition 15, $Z_2(N) \leq A$, and $Z_2(F) \leq N$, therefore by Proposition 18, $A/Z_2(N) \leq^{c} N/Z_2(N)$ and $N/Z_2(F) \leq^{c} F/Z_2(F)$, since by[8] $Z_2(N) = N \cap Z_2(F) = Z_2(F)$, then $A/Z_2(F) \leq^{c} N/Z_2(F)$ and $N/Z_2(F) \leq^{c} F/Z_2(F)$, therefore by[9] $A/Z_2(F) \leq^{c} F/Z_2(F)$, so by Proposition 19, $A \leq^{c} F$. If $A \leq^{tes} F' \leq F$, $A \cap B=0$, for some $B \leq F'$, then $A \cap B \leq Z_2(F)$, hence $B \leq Z_2(F) \leq A$, then $B = A \cap B = 0$, so $A \leq^{e} F'$, contradiction $A \leq^{c} F$, therefore $A \leq^{tc} F$.

Remark 27: For any semimodule F. If $A \leq^{c} F$ and $A' \leq^{c} F$, this does not lead to $A \cap A' \leq^{c} F$.

Note that: every module is a semimodule and every direct summand is closed .In [12] example 1 and 3 show that the intersection of two closed in F is not necessarily closed in F.

Proposition 28: Let F be an R-semimodule if $A \leq F$ and $A' \leq^{tc} F$, then $A \cap A' \leq^{tc} A$.

Proof: Assume that $A \cap A' \leq^{tes} D \leq A$, then by Propositions 11 and 13, $D/(A \cap A')$ is Z_2 torsion, hence $D/(D \cap A')$ is Z_2 torsion (since $D/(D \cap A')$ is homomorphic image of $D/(A \cap A')$). But $D/(D \cap A') \cong (D + A')/A'$, therefore $(D + A')/A'$ is Z_2 torsion, so by Proposition 12, $A' \leq^{tes} (D + A')$ a contradiction, then $A' = D + A'$, and $D \leq A'$. But $A \cap A' \leq D$, then $D = A \cap A'$, and so $A \cap A' \leq^{tc} A$.

Proposition 29: Let F be a cancellative semi subtractive R-semimodule. An arbitrary intersection of t-closed subtractive subsemimodule is t-closed.

Proof: Assume that $C = \bigcap_{\lambda \in \Lambda} C_\lambda$, where C_λ is a t-closed subsemimodule of F, for any λ in index set Λ. Let $\theta: F/C \to \prod_\lambda (F/C_\lambda)$, defined by $m + C \mapsto (m + C_\lambda)$. If $m+C=m'+C$, then $m+c_1=m'+c_2$, where $c_1,c_2 \in C$, hence c_1, $c_2 \in C_\lambda$, so for each λ, $(m + C_\lambda) = (m' + C_\lambda) \in \prod_\lambda (F/C_\lambda)$, therefore θ is well defined . Now let $(m + C_\lambda) = (m' + C_\lambda)$, then $m + C_\lambda = m' + C_\lambda$, so for each λ , $m + c_\lambda = m' + c'_\lambda$, where $c_\lambda, c'_\lambda \in C_\lambda$ for each $\lambda \in \Lambda$. By semi subtractive there exists t such that either $m+t=m'$ or $m=m'+t$.
Case one: $m+t=m'$, by cancellative, $c_\lambda = t + c'_\lambda$, so by subtractive $t \in C_\lambda$, hence for each λ, $t \in C$.
Case two: $m=m'+t$, by cancellative, $c_\lambda + t = c'_\lambda$, by subtractive $t \in C_\lambda$, hence for each λ, $t \in C$. Therefore $m+C=m'+C$, hence θ is monomorphism. Since F/C_λ is nonsingular [by Proposition 21], then $\prod_\lambda (F/C_\lambda)$ is nonsingular, and hence F/C is nonsingular. So $C \leq^{tc} F$.

III. T-EXTENDING SEMIMODULE

In this section, the t-extending semimodule is introduced and investigated. A number of properties of the t-extending semimodule are also studied by proving the equivalent statement to this concept.

Definition 30: A semimodule F is said to be t-extending if every t-closed subsemimodule is a direct summand.

Recall, a semimodule F is called *extending* if every subsemimodule of F is essential in a direct summand of F. Equivalently, every closed subsemimodule of F is a direct summand of F [9].

Remark 31:

1. Every Z_2- torsion semimodule is t- extending.
2. Every extending semimodule is t- extending.

***Proof*:(1)**, Let F be Z_2- torsion, then the only t- closed subsemimodule of F is F which is a direct summand of F, then F is t- extending.

(2), Assume F is extending and let $C \leq^{tc} F$, then $C \leq^{c} F$ [by Remark 4(3)], since F is extending, C is a direct summand of F, so F is t- extending.

Note that: For example to (2), Z_6 is extending and t-extending semimodules. But the inverse of (2) is not true. For example where $F=\mathbb{Z}_8 \oplus \mathbb{Z}_2$, then F is singular, hence each subsemimodule is t- essential, therefore F is t-extending , but by [9], F is not extending.

Proposition 32: Let F be an R–semimodule. If F is t-extending then for any subsemimodule A of F, A_2 is a direct summand whenever $A_2 / A = Z_2(F/AA)$.

Proof: Since $(F/A_2) \cong (F/A)/(A_2/A) = (F/A)/Z_2(F/A)$, then by Proposition 21 $A_2/A \leq^{tc} F/A$ therefore by Proposition 16, $A_2 \leq^{tc} F$, since F is t- extending then A_2 is a direct summand of F.

Proposition 33: Let F be an R–semimodule. If F is t-extending then $F= Z_2(F) \oplus F'$, where F' is nonsingular extending semimodule.

Proof: Since by Remark 4(7), $Z_2(F) \leq^{tc} F$, and F is t-extending then $Z_2(F)$ is a direct summand of F , say $F= Z_2(F) \oplus F'$, for some $F' \leq F$, hence F' is nonsingular (since $F/Z_2(F) \cong F'$, and $F/Z_2(F)$ is nonsingular). Let $C \leq^{c} F'$, since F' is nonsingular then by Remark 4(4), $C \leq^{tc} F'$ so by Proposition 21, F'/C is nonsingular. Since $C \leq Z_2(F) \oplus C$, then $F'/Z_2(F) \oplus C$ is nonsingular, that is $Z_2(F) \oplus C \leq^{tc} F$, therefore $Z_2(F) \oplus C$ is a direct summand of F(since F is t-extending), say $F=Z_2(F) \oplus C \oplus F''$, by Semi modular law[1] $F'=C \oplus (Z_2(F) \oplus F'') \cap F'$, so C is a direct summand of F', and F' is extending.

Proposition 34: Let F be a subtractive R–semimodule. If F is t-extending then every subsemimodule of F containing $Z_2(F)$ is essential in a direct summand of F.

Proof: Let $A \leq F$ such that $Z_2(F) \leq A$, since F is t – extending then $Z_2(F)$ is a direct summand of F, say $F= Z_2(F) \oplus F'$, for some $F' \leq F$, then by Semi modular law $A=Z_2(F) \oplus (F' \cap A)$, since $(F' \cap A) \leq F'$ and F' is extending by Proposition 33, then there exists a direct summand L of F', such that $F' = L \oplus F''$ for some $F'' \leq F'$ and $(F' \cap A) \leq^{e} L$, therefore, $A = Z_2(F) \oplus (F' \cap A) \leq^{e} Z_2(F) \oplus L$, where $Z_2(F) \oplus L$ is a direct summand of F (since $F = Z_2(F) \oplus L \oplus F''$).

Proposition 35: Let F be an R–semimodule. If F is t-extending then every subsemimodule of F is t-essential in a direct summand of F.

Proof: Let $A \leq F$, then by 32, $Z_2(F/A) = N/A$ where N is a direct summand of F, hence N/A is Z_2 torsion since $Z_2(N/A) = (N/A) \cap Z_2(F/A) = N/A$), therefore by Proposition 12, $A \leq^{tes} N$.

Proposition 36: Let F be an R –semimodule. Then F is t-extending if and only if for every subsemimodule A of F there exists a decomposition $F/A = N/A \oplus N'/A$, such that N is a direct summand of F, and $N' \leq^{tes} F$.

Proof: Let $A \leq F$, then by Proposition 35 there exists a decomposition $F = N \oplus L$, such that $A \leq^{tes} N$, then $F/A = N/A \oplus ((L \oplus A)/A)$, since $F/(L \oplus A) \cong (F/A)/(L \oplus A)/A) \cong N/A$, but by Propositions 11 and 13, N/A is Z_2 torsion, so $F/(L \oplus A)$ is Z_2 torsion, therefore by Proposition 12, $L \oplus A \leq^{tes} F$, when $N' = L \oplus A$, hence $N' \leq^{tes} F$. Conversely, let $A \leq^{tc} F$, then by assumption there exists a decomposition $F/A = N/A \oplus N'/A$, since

$F/N' \cong N/A$, hence N/A is singular, and therefore $A \leq^{tes} N$ by Lemma 7 and Proposition 12, a contradiction , then $A = N$, and so A is a direct summand of F, that is F is t-extending.

Proposition 37: Every homomorphic image of t-extending semimodule is t-extending.

Proof: Let F be a t-extending semimodule. It is enough to show that F/A is t-extending for any subsemimodule A of F. Let $L/A \leq F/A$, since F is t-extending, then by Proposition 35 there exists a direct summand N of F such that $F = N \oplus F'$, for some $F' \leq F$, and $L \leq^{tes} N$, then by Propositions 13 and 11, N/L is Z_2-torsion, but $N/L \cong (N/A)/(L/A)$, then by Proposition 12, $L/A \leq^{tes} N/A$, hence F/A is t- extending.

Through the previous Proposition, the following results can be obtained:

Corollary 38: Every direct summand of t-extending is t-extending.

Proof: It is clear by Proposition 37.

Proposition 39: Every direct sum of t-extending semimodule is t-extending.

Proof: Assume that $F = F_1 \oplus F_2$, where F_1 and F_2 are t-extending, and let $A \leq^{tc} F$. Let $\pi_i: F \to F_i$ be the natural projections from F onto F_i (i=1, 2), then $A = \pi_1(A) \oplus \pi_2(A)$, since F_1 and F_2 are t-extending, then there exists direct summand D_1 and D_2 of F_1 and F_2 respectively, such that $\pi_1(A) \oplus \pi_2(A) \leq^{tes} D_1 \oplus D_2$, but $D_1 \oplus D_2$ is a direct summand of F (since $F_1 = D_1 \oplus D_1{}'$ and $F_2 = D_2 \oplus D_2{}'$, therefore $F = D_1 \oplus D_1' \oplus D_2 \oplus D_2' = (D_1 \oplus D_2) \oplus (D_1{}' \oplus D_2{}')$, then F is t-extending .

A semimodule F is said to be ***semisimple*** if it is a direct sum of its simple subsemimodules [13].

Corollary 40: Let F_1 be a semisimple R- semimodule, then $F = F_1 \oplus F_2$ is t-extending for any t-extending F_2.

Proof: Since F_1 and F_2 are t-extending then by Proposition 39, F is t-extending.

Proposition 41: Let $F = F_1 \oplus F_2$, be a nonsingular subtractive semimodule, then F is t-extending if and only if every t- closed $K \leq F$ with $K \cap F_1 = 0$ or $K \cap F_2 = 0$ is a direct summand.

Proof: ($\Longrightarrow$) Let F be t-extending, and let $K \leq^{tc} F$, such that $K \cap F_1 = 0$, then by assumption, there exists a direct summand N of F such that $K \leq^{tes} N$, a contradiction, then $K=N$, and similarly when $K \cap F_2 = 0$.

($\Longleftarrow$) Let $B \leq^{tc} F$ then either $B \cap F_1$=0, then by assumption B is direct summand of F. Or $B \cap F_1 \neq 0$, then there exists D such that $B \cap F_1 \leq^{tes} D \leq^{tc} B$ (by [9], Proposition 10 and Remark 4(4), then $D \cap F_2 = 0$(since $B \cap F_1 \cap D \cap F2 = 0$. Note that $D \leq^{tc} F$ by Proposition 26, then by assumption, D is a direct summand of F, that is, $F = D \oplus D'$ for some $D' \leq F$, by Semi modular Law, $B = D \oplus (B \cap D')$, but $(B \cap D')$ is t-closed in F, then $(B \cap D') \cap F_2 = 0$, also by assumption $(B \cap D')$ is a direct summand of D', then $D' = (B \cap D') \oplus D''$ for some $D'' \leq D'$, so $F = D \oplus (B \cap D') \oplus D'' = B \oplus D''$, therefore B is a direct summand of F and F is t-extending.

A semimodule F is said to be ***uniform*** if any subsemimodule N of F is essential [14].

Remark 42: Every semisimple (uniform) R-semimodule is t-extending.

Proof: Assume F is a semisimple or uniform R–semimodules, then F is extending R-semimodules so by Remark 31(2) F is t-extending.

A subsemimodule A of F is said to be ***fully invariant*** if $f(A) \subseteq A$ for each R-endomorphism f on F [15].

Proposition 43: Every fully invariant subsemimodule of t-extending is t-extending.

Proof: Let F be t-extending and N be a fully invariant subsemimodule of F , and let $A \leq N$, then $A \leq F$, since F is t-extending then there exists a direct summand F' of F, say $F = F' \oplus F''$ such that, $A \leq^{tes} F'$, since N is a fully invariant then by[16], $N = N \cap F' \oplus N \cap F''$. Clearly $A \leq^{tes} N \cap F'$(since$N \cap F' \leq F'$), hence by Proposition 35, N is t-extending.

IV. CONCLUSION

This work presented the t-extending semimodule, thereby discussing the t-essential and t-closed properties as pre-concepts. It is shown that in the nonsingular semimodules, the t-essential and essential properties are equivalent. The t-closed property is closed under factor. For proving that the invers image of a t-closed subsemimodule to be t-closed, extra conditions were required, such as semisubtractive and cancellative semimodule, with subtractive subsemimodule. It is shown that the t-extending property is closed under homomorphic image (factor) hence direct summand, while under direct sum it needs some extra conditions.

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