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TRIGONOMETRIC JACKSON INTEGRALS APPROXIMATION BY A k-GENERALIZED MODULUS OF SMOOTHNESS

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Abstract. The need for smoothness measures emerged by mathematicians working in the fields of approximation theory, functional analysis and real analysis. In the present paper, a new version of generalized modulus of smoothness is studied. The aim of defining that modulus, is to find the degree of best L_p functions approximation via trigonometric polynomials. We benefit from Jackson integrals to arrive to the essential approximation theorems.

1. Introduction

Many papers have been introduced about approximation theorems in terms of moduli of smoothness (see [2, 3, 4, 6]). In addition, numerous authors have investigated the problem of approximation by several types of functions, see [5].

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In [6], the authors proved direct and inverse inequalities for the best algebraic approximation with modulus of smoothness with order 2.

Here, we define kth order modulus of smoothness to serve as a generalization of their modulus.

2. Preliminaries

Now let us list some definitions, which are used throughout our present paper. Let the following periodic function space is given by

$$L_p[-\pi,\pi]=\left\{f:\|f\|_p=\left(\int_{-\pi}^{\pi}|f(x)|^p dx\right)^{1/p}<\infty\right\}.$$

We define here an operator that yeilds a version of trigonometric Jackson integrals as follow,

Definition 2.1. For any $m,k\in N,\ f\in L_p[-\pi,\pi]$, define

$$G_{m,k}(f(x))=\sum_{i=1}^{k} f\left(\cos\left(\arccos x+\left(\frac{k}{2}-i\right)h\right)\right)J_m(x),$$

where $J_m(x)=\frac{1}{2}+\sum_{i=1}^{k}\rho_{i,m}\cos ix$, is the Fejer Kernel from [1], that satisfies

$$\int_{-\pi}^{\pi}J_m(x)dx=1.$$

Now, define

$$\Omega_k(f,t)_p=\sup_{|h|\le t}\|\Delta_h^k f(\cdot)\|_p,$$

to be the k-genalized modulus of smoothness in the interval $[-\pi,\pi]$, where

$$\Delta_h^k f(x)=\sum_{i=1}^{k}\binom{k}{i}(-1)^k f\left(\cos\left(\arccos x+\left(\frac{k}{2}-i\right)h\right)\right).$$

Define Ξ to be the set of all operators of the above form of $G_{m,k}$.

Define,

$$E_n(f)_p=\inf_{G\in\Xi}\|f-G\|_p$$

to be the degree of best approximation of $L_p[-\pi,\pi]$ out of Ξ.

3. Main results

We introduce the results that we need for the proof of our approximation theorems.

Lemma 3.1. *For any* $G_{m,k} \in \Xi$ *, we have*

$$\Omega_{k,r}^{\varphi}\left(G_{m,k},t\right)_p \le C\|G_{m,k}\|_p,$$

where C *is a constant of* p, m *and* k.

Proof. It is direct from quasi-triangle inequality that

$$\begin{aligned}
&\left\|\Delta_h^k G_{m,k}(f)\right\|_p^p \\
&= \left\|\sum_{i=1}^{k}\binom{k}{i}(-1)^k f\left(\cos\left(\arccos x+\left(\frac{k}{2}-i\right)h\right)\right)J_m(x)\right\|_p^p \\
&\le C\sum_{i=1}^{k}\binom{k}{i}(-1)^k \left\|f\left(\cos\left(\arccos x+\left(\frac{k}{2}-i\right)h\right)\right)J_m(x)\right\|_p^p \\
&\le C\left\|f\left(\cos\left(\arccos x+\left(\frac{k}{2}-i\right)h\right)\right)J_m(x)\right\|_p^p \\
&\le C\left(m,k,p\right)\left\|G_{m,k}(f)\right\|_p^p.
\end{aligned}$$

□

Also, we need the following lemma (see [2]).

Lemma 3.2. *Let* $I = \cup I_m, m = 1, \cdots, n$, *where* $|I_m| \sim h \sim \frac{1}{n}$ *be any partition of* I. *Then for any* $m < n$, *and* $f \in L_p$,

$$\sum_{m=1}^{n} E_m(f)_{L_p(I_m)} \le CE_n(f)_{L_p(I)}.$$

The generalized modulus of smoothness that defined in Definition 2.1 has the following properties.

Theorem 3.3. *For any* $f \in L_p[-\pi,\pi]$, *we have*

(1) $\lim_{t\to 0}\Omega_k\left(f,t\right) = 0$,
(2) $\Omega_k\left(f+g,t\right)_p \le \Omega_k\left(f,t\right)_p + \Omega_k\left(g,t\right)_p$,
(3) $\Omega_k\left(f,t\right)_p \le \Omega_k\left(f,t'\right)_p$, *for any* $t \le t'$,
(4) $\Omega_k\left(f,nt\right)_p \le n^k\Omega_k\left(f,t\right)_p$, *for any* $n \in N$,
(5) $\Omega_k\left(f,\lambda t\right)_p \le \left(1+\lambda\right)^k\Omega_k\left(f,t\right)_p$, $\lambda \in R$,
(6) $\Omega_k\left(f,t\right)_p \le C\|f\|_p$.

Proof. Proofs of properties (1), (2) and (3) follow directly from definition of generalized moduli of smoothness. To prove property (4), we use the difference identity from [2]:

$$\Delta_{nh}^{k} = \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{nh}^{k} f\left(x + \left(i_1 + i_2 + \cdots + i_k - \frac{nk}{2}\right) h\right).$$

We get the result by induction on k. From definition of kth difference and operator G, we have

$$\Delta_{nh} f(x) = \sum_{i=0}^{n-1} f\left(cos\left(\arccos x + \left(\frac{k}{2} - i\right) h\right)\right).$$

Suppose that the relation is true for k. Then

$$\begin{aligned}\Delta_{nh}^{k+1} f(x) &= \Delta_{nh}^{k}[\Delta_{nh} f(x)] \\ &= \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Delta_{nh}^{k} f\Big(\cos(\arccos x + (\frac{k}{2} - i_1)h + \cdots \\ &\qquad + (\frac{k}{2} - i_k)h + (\frac{k+1}{2} - i_{k+1})h)\Big).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\| \Delta_{nh}^{k+1} f(x) \|_p^p &\leq \sum_{i_1=0}^{n-1} \cdots \sum_{i_k=0}^{n-1} \Big\| l\Delta_{nh}^{k} f\Big(\cos(\arccos x + \left(\frac{k}{2} - i_1\right) h + \cdots \\ &\qquad + \left(\frac{k}{2} - i_k\right) h + \Big(\frac{k+1}{2} - i_{k+1}\Big)h)\Big)\Big\|_p^p \\ &= n^k \Omega_k(f,t)_p^p.\end{aligned}$$

Moreover, we let $n-1 < \lambda \leq n$, to prove (5). Then

$$\Omega_k(f, \lambda t)_p \leq \Omega_k(f, nt)_p \leq n^k \Omega_k(f, nt)_p \leq (\lambda + 1)^k \Omega_k(f,t)_p.$$

For (6), it is easily to be proved by using inequalities of quasi-normed space, as follow:

$$\begin{aligned}\Omega_k(f,t)_p^p &\leq 2^p \sum_{i=0}^{k} \binom{k}{i}^p \int_{-\pi}^{\pi} \sup_{h \leq t} \left| f\left(\cos\left(\arccos x + \left(\frac{k}{2} - i\right) h\right)\right)\right|^p dx \\ &\leq C(k,p) \|f\|_p^p.\end{aligned}$$

□

It is time to find and estimate the degree of best approximation in the following theorems.

Theorem 3.4. *For any* $f \in L_p[-\pi, \pi]$, $0 < p < 1$, *we have*

$$E_n(f)_p \le C\Omega_k\left(f, \frac{1}{n}\right)_p.$$

Proof. Let $f \in L_p[-\pi, \pi]$, $G_{m,k} \in \Xi$. By using Lemma 2.1, we have

$$\begin{aligned}
&E_n(f)_p \\
&\le \|f - G_{n,k}(f)\|_p^p \\
&= \int_{-\pi}^{\pi} \left| \sum_{i=1}^{k} \binom{k}{i} (-1)^k f\left(\cos\left(\arccos x + \left(\frac{k}{2} - i\right) h\right)\right) J_m(x) - f(x) \right|^p dx \\
&\le C \sum_{i=1}^{k} \binom{k}{i} (-1)^k \int_{-\pi}^{\pi} \left| f\left(\cos\left(\arccos x + \left(\frac{k}{2} - i\right) h\right)\right) - f(x) \right|^p \\
&\qquad\qquad \times \int_{-\pi}^{\pi} |J_m(x)|^p \, dx \\
&\le C\Omega_k(f, h)_p^p.
\end{aligned}$$

□

To make the degree of approximation more accurate, we prove the following lower estimate, so that the degree of approximation is essentially estimated.

Theorem 3.5. *For any* $f \in L_p[-\pi, \pi]$, $0 < p < 1$, *we have*

$$\Omega_k\left(f, \frac{1}{n}\right)_p \le C(p) E_n(f)_p.$$

Proof. For $m < n$, suppose that

$$\|G_n - G_m\|_p \le 2C(p) E_m(f)_p.$$

Now, by letting $b = \max\{i : 2^{-i} < n\}$, so that by Lemma 2.1, Lemma 2.2 and Theorem 3.4 we get

$$\begin{aligned}
\Omega_k(f, t)_p^p &\le C\left[\Omega_k(f - G_n, t)_p^p + \Omega_k(G_n, t)_p^p\right] \\
&\le C\left[\|f - G_n\|_p^p + \Omega_k\left(G_{2^i} - G_{2^{i-1}}, 2^{-b}\right)_p^p\right] \\
&\le C\left[\Omega_k(f, t)_p^p + \sum_{m=2}^{n} (m+1)^{p-1} E_m(f)_{L_p(I_m)}^p\right] \\
&\le 2CE_n(f)_p^p.
\end{aligned}$$

□

4. Conclusions

Studying inverse and direct theorems is very important in the field of function approximation theory. In L_p spaces, we estimated the degree of best approximation by using newly defined trigonometric operators via generalized modulus of smoothness. Thus more generalizations are done for the main theorems.

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