On Asymptotic Distributions Theory

Dr. Kareema Abdul-Kaddum
Department of Mathematics
College of Education

Jinan Hamzah Farhood
Department of Mathematics
College of Education

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Abstract

The essence of asymptotic methods is approximation. The main object of this thesis is to give a unified derivation of some results and theorems.

Also, this research deals with asymptotic distributions that is the distributions we obtained by letting the time horizon (sample size) tends to infinity. The research methodology is theoretical. We obtain some results for the univariate case (for example about sequence of random variables) and the multivariate case (for example about sequence of random vectors) about the asymptotic theory.

حول نظرية التوزيعات التقاربية

د. كريمة عبد الكاظم
قسم الرياضيات – كلية التربية

جنان حمزة فرهود
قسم الرياضيات – كلية التربية

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الخلاصة

ان جوهر الطرق التقاربية هو التقريب وان الهدف الرئيسي من هذه الرسالة هو إعطـــاء اشـــتقاق موحد لبعض النتائج والنظريات.

إن هذا البحث يتعامل أيضا مع التوزيعات التقاربية التي نحصل عليها عندما يقترب حجم العينة من المالانهاية (بتزايد بصورة غير متناهية)

ولقد أوجدنا بعض النتائج لحالة المتغير الواحد (على سبيل المثال متتابعة المتغيرات العشوائية) وحالة أكثـــر من متغير (على سبيل المثال متتابعة المتجهات العشوائية) حول نظرية التقارب.

Introduction

Exact distribution theory is limited to very special cases (normal independent identically distributed (i.i.d.) errors linear estimators), or involves very difficult calculations. This is too restrictive for applications. By making approximations based on large sample sizes, we can obtain distribution theory that is applicable in a much wider range of circumstances. These approximations are sometimes quite accurate and can often be constructed without a complete specification of the population distribution for the data. Suppose $F_n(x)$ is the (unknown) cumulative distribution function for some statistic based on a sample of size *n*. If it can be shown that the sequence of functions $F_1(x), F_2(x)$, … converges rapidly to a known limit $F(x)$ as *n* tends to infinity, then we might use $F(x)$ as an approximation to $F_n(x)$ even for moderate values of *n*. The quality of the approximation depends on the speed of convergence, but can be checked by computer simulation.

The simplest example of this approach `is the average of independent draws from a distribution possessing a finite variance. Let $\overline{X}_n = n^{-1}\sum_{i=1}^{n} X_i$, where the X's are i.i.d. with population mean $= E(X_i) = \mu$ and population variance $= \mathrm{Var}(X_i) = \sigma^2$. By an easy calculation, we find that $\overline{X}_n$ has mean μ and variance σ^2 / n. Although the exact distribution of $\overline{X}_n$ depends on the distribution of X's, a simple asymptotic approximation is always available. The cumulative distribution function F_n for $\overline{X}_n$ is quite sensitive to the value of *n* so we would not expect the limit of the sequence F_1, F_2,… to yield a good approximation to F_n unless *n* is very large. But the standardized random variable $S_n = \sqrt{n}(\overline{X}_n - \mu)/\sigma$ has mean zero and variance one for every *n*; its cumulative distribution function, say F_n^*, is much less sensitive to the value of *n*. Thus, if we could find the limit F^* of the sequence F_1^*, F_2^*, …, we might be willing to use it as an approximation to the distribution of S_n. The sequence F_1^*, F_2^*, … necessarily converges to the standard normal cumulative distribution function. This leads us to approximate F_n by the cumulative distribution function of a $N(\mu, \sigma^2 / n)$ distribution.

People may ask; since asymptotic distribution is only an approximation, why we are not using the exact distribution instead? Unfortunately, the exact finite sample distribution in many cases are too complicated to derive, even for Gaussian processes. Therefore, we use asymptotic distribution as alternatives.

Most econometric methods used in applied economics are asymptotic in the sense that they are likely to hold only when the sample size is “large enough”.

Asymptotic theory involves generalizing the usual notions of convergence for real sequences to allow for random variables. It is important to emphasis that limiting distributions obtained by central limit theorem (CLT) all involve unknown parameters which we seek to estimate.

The asymptotic distribution was studied by other researchers who worked in our field which as the following :

The asymptotic distribution of the likelihood ratio test that is one technique for detecting a shift in the mean of a sequence of independent normal random variables derived by (Irvine,1986). The empirical process proof of the asymptotic distribution of sample quantiles studied by (Rust,1998).

The asymptotic distribution of gumbel statistic in a semi-parametric approach (*) derived by (Alves,1999) where this note is an answer to some open problems connected with recent developments for appropriate methodologies for making inferences on the tail of a distribution function (d.f.).The asymptotic results for the linear regression model studied by (Flinn,1999). A general result concerning the large sample distribution of Moran I type test statistic given by (Kelejian & prucha,1999) and applied this result to derive the large sample distribution of the Moran I test statistic for a variety of important models for which general spatial correlation testing procedures are not available.

A review of basic elements of asymptotic theory provided by (potscher & prucha,1999) . Topics included modes of convergence, laws of large numbers and central limit theorems.

The asymptotic distribution of the Euclidean distance between MA models studied by(Sarno, 2001). An introduction to asymptotic oncepts in statistics studied by (weeks,2003).An introduction to asymptotic concepts in statistics studied by (Weeks,2003) .

The asymptotic distribution of a simple two-stage (Hannan-Rissanen-type) linear estimator for stationary invertible vector autoregressive moving average

(VARMA) models in the echelon form representation studied by(Dufour & Jouini,2004) . General conditions for consistency and asymptotic normality are given. A consistent estimator of the asymptotic covariance matrix of the estimator is also provided, so that tests and confidence-intervals can easily be constructed.

The formal properties of corrrelators of eigenvalues in the so-called planar limit (semiclassical) of various matrix models in terms of certain algebra-geometric data presented by (Bertola,2004). The asymptotic distribution of a set of linear restrictions on regression coefficients studied by (Anderson,2004) where reduced rank regression analysis provided maximum likelihood estimators of a matrix of regression coefficients of a specified rank and of corresponding linear restrictions on such a matrix. These estimators depended on the eigenvectors of an “effect” matrix in the metric of an error covariance matrix and shown that the maximum likelihood estimator of the restrictions can be approximated by a function of the effect matrix alone. The procedures are applied to a block of simultaneous equations. The block may be over-identified in the entire model and the individual equations just-identified within the block.

Definition (1) (Fahady &Shamoon,1990)

Consider distribution functions $F_1(.)$, $F_2(.)$, … and $F(.)$. Let X_1, X_2, … and X denote random variables (not necessarily on a common probability space) having these distributions, respectively. We say that X_n *converges in distribution* (or in law) to X if

$$\lim_{n\to\infty} F_n(v) = F(v), \text{ for all } v \text{ which are continuity points of F.}$$

This is written $X_n \xrightarrow{d} X$ or $X_n \xrightarrow{L} X$ or $F_n \xrightarrow{w} F$.

The limiting distribution function, F, is referred to as the asymptotic distribution of X_n, and provides the basis for approximating the distribution of X_n, as n increases without bounds.

In practice when the mean or variance of X_n increase with n, in deriving the asymptotic distribution of X_n it is necessary to consider the limiting distribution of normalized or rescaled random variable, $Z_n = \dfrac{X_n - \mu_n}{\sigma_n}$, where μ_n and σ_n are appropriate constants.

In general, we would like to say that the distribution of the random variables X_n converges to the distribution of X if

$F_n(x) = P(X_n < x) \to F(x) = F(X < x)$ for every $x \in R$.

Theorem (2) (Serfling,1980)

Let the distribution functions F, F_1, F_2, … possess respective characteristic functions ϕ, ϕ_1, ϕ_2, … . The following statements are equivalent:

(i) $F_n \xrightarrow{w} F$ (or $X_n \xrightarrow{d} X$);

(ii) $\lim_{n\to\infty} \phi_n(\theta) = \phi(\theta)$, for each real θ;

(iii) $\lim_{n\to\infty} \int g \, dF_n = \int g \, dF$, for each bounded continuous function g.

Proposition (3) (Dufour,2003)

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of random variables such that $X_n - Y_n \xrightarrow{p} 0$ and $Y_n \xrightarrow{d} Y$, and let $g : R \to R$ be a continuous function. Then

(a) $X_n \xrightarrow{d} Y$;

(b) $g(X_n) - g(Y_n) \xrightarrow{p} 0$;

(c) $g(X_n) \xrightarrow{d} g(Y)$.

Definition (4) (Boik,2004)

Let $\{X_n\}$ be a sequence of random variables. $X_n = o_p(1)$ if $X_n \xrightarrow{p} 0$. That is, for every $\varepsilon > 0$, $\lim_{n\to\infty} P(|X_n| < \varepsilon) = 1$

or, equivalently, for every $\varepsilon > 0$ and for every $\eta > 0, \exists$ an integer $n(\varepsilon,\eta)$

Such that if $n > n(\varepsilon,\eta)$ then

$$P(|X_n| < \varepsilon) \geq 1 - \eta .$$

One can say, informally, that $X_n = o_p(1)$ if $X_n = o(1)$ with arbitrarily high probability.

1. Some results for a univariate case about the asymptotic theory

Lemma (5)

Let $\{X_n\}$ be a sequence of independent random variables such that $X_{ni} \xrightarrow{d} X$, $i = 1, \ldots, m$, where X is a random variable. Then $\sum_{i=1}^{m} X_{ni} \xrightarrow{d} mX$.

Proof:

Let $\phi_n(\theta)$ be the characteristic function of $\sum_{i=1}^{m} X_{ni}$, for any real θ. Then

$$\phi_n(\theta) = E\left(e^{i\theta \sum_{i=1}^{m} X_{ni}} \right)$$

and

$$\lim_{n\to\infty} \phi_n(\theta) = E\left(e^{i\theta m X}\right)$$

which is the characteristic function of mX and this is implies that

$$\sum_{i=1}^{m} X_{ni} \xrightarrow{d} mX \text{. (by Theorem 2(i),(ii)).}$$

Theorem (6)

Let $\{X_n\}$ be a sequence of independent random variables and $\{Y_n\}$ be a sequences of random variables such that $X_{ni} \xrightarrow{d} X$, $i = 1, \ldots, m$ and $Y_n \xrightarrow{p} c$, where X is a random variable and c is a constant not infinity. Then

(a) $\sum_{i=1}^{m} X_{ni} \pm Y_n \xrightarrow{d} mX \pm c$;

(b) $\sum_{i=1}^{m} X_{ni} Y_n \xrightarrow{d} mcX$;

(c) $\dfrac{\sum_{i=1}^{m} X_{ni}}{Y_n} \xrightarrow{d} \dfrac{mX}{c}$ if $c \neq 0$.

Proof

(a) Choose and fix v such that v–c is a continuity point of $F_{mX}(v)$. Let $\varepsilon > 0$ be such that $v - c + \varepsilon$ and $v - c - \varepsilon$ are also continuity points of $F_{mX}(v)$. Then

$$F_{\sum_{i=1}^{m} X_{ni} + Y_n}(v) = P\left(\sum_{i=1}^{m} X_{ni} + Y_n \le v\right)$$

$$\le P\left(\sum_{i=1}^{m} X_{ni} + Y_n \le v, | Y_n - c | < \varepsilon\right) + P(| Y_n - c | \ge \varepsilon)$$

$$\le P\left(\sum_{i=1}^{m} X_{ni} \le v - c + \varepsilon\right) + P(| Y_n - c | \ge \varepsilon).$$

Hence by the hypotheses of the theorem, and by the choice of $v - c + \varepsilon$,

$$\limsup_{n\to\infty} F_{\sum_{i=1}^{m} X_{ni} + Y_n}(v) \le \limsup_{n\to\infty} P\left(\sum_{i=1}^{m} X_{ni} \le v - c + \varepsilon\right) + \limsup_{n\to\infty} P(| Y_n - c | \ge \varepsilon)$$

$$= F_{mX}\,(v - c + \varepsilon).$$

Since $Y_n \xrightarrow{p} c$, we have $\limsup_{n\to\infty} P(| Y_n - c | \ge \varepsilon) = 0$.

Similarly,

$$P\left(\sum_{i=1}^{m} X_{ni} \le v - c - \varepsilon\right) \le P\left(\sum_{i=1}^{m} X_{ni} + Y_n \le v\right) + P(| Y_n - c | \ge \varepsilon)$$

and thus

$$F_{mX}(v - c - \varepsilon) \le \liminf_{n\to\infty} F_{\sum_{i=1}^{m} X_{ni} + Y_n}(v).$$

Since v – c is a continuity point of $F_{mX}(v)$, and since ε may be taken arbitrarily small, we have

$$\lim_{n\to\infty} F_{\sum_{i=1}^{m} X_{ni} + Y_n}(v) = F_{mX}(v - c) = F_{mX+c}(v).$$

This is follows that

$$\sum_{i=1}^{m} X_{ni} + Y_n \xrightarrow{d} mX + c \text{ (by Definition (1)).}$$

And to proof the other direction, we have

Let $\phi_n(\theta)$ be the characteristic function of $\sum_{i=1}^{m} X_{ni} - Y_n$, for any real θ.

Then

$$\phi_n(\theta) = E\left(e^{i\theta\left(\sum_{i=1}^{m} X_{ni} - Y_n\right)}\right) = e^{-i\theta Y_n} E\left(e^{i\theta \sum_{i=1}^{m} X_{ni}}\right)$$

and

$$\lim_{n\to\infty} \phi_n(\theta) = e^{-i\theta c} E(e^{i\theta m X})$$

which is the characteristic function of mX–c and this is implies that

$$\sum_{i=1}^{m} \mathrm{X}_{ni} - \mathrm{Y}_n \xrightarrow{d} m\mathrm{X} - \mathrm{c} \text{ (by theorem (2)(i),(ii)).}$$

(b) Let $Z_n = \sum_{i=1}^{m} \mathrm{X}_{ni}(\mathrm{Y}_n - c)$, and for arbitrary positive constants ε and δ, consider

$$P(|\mathrm{Z}_n| > \varepsilon) = P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| |\mathrm{Y}_n - \mathrm{c}| > \varepsilon, |\mathrm{Y}_n - \mathrm{c}| < \frac{\varepsilon}{\delta}\right) +$$

$$P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| |\mathrm{Y}_n - \mathrm{c}| > \varepsilon, |\mathrm{Y}_n - \mathrm{c}| \geq \frac{\varepsilon}{\delta}\right)$$

$$\leq P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| \geq \delta\right) + P\left(|\mathrm{Y}_n - \mathrm{c}| \geq \frac{\varepsilon}{\delta}\right).$$

For any fixed δ, taking limits of both sides of the above inequality, and noting that by assumption $\mathrm{Y}_n \xrightarrow{p} c$ and $\sum_{i=1}^{m} \mathrm{X}_{ni} \xrightarrow{d} m\mathrm{X}$, we have

$$\lim_{n\to\infty} P(|\mathrm{Z}_n| > \varepsilon) \leq \lim_{n\to\infty} P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| > \delta\right) = P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| > \delta\right).$$

But δ is arbitrary and hence $P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| > \delta\right)$ can be made as small as desired by choosing a large enough value for δ. Therefore

$\lim_{n\to\infty} P(|\mathrm{Z}_n| > \varepsilon) = 0$ and $\mathrm{Z}_n \xrightarrow{P} 0$. Hence by proposition (3)(*a*),

$\sum_{i=1}^{m} \mathrm{X}_{ni}\mathrm{Y}_n$ and $\mathrm{c}\sum_{i=1}^{m} \mathrm{X}_{ni}$ will have the same asymptotic distribution given by the distribution of mcX.

(c) Let $\mathrm{Z}_n = \sum_{i=1}^{m} \mathrm{X}_{ni}\left(\frac{1}{\mathrm{Y}_n} - \frac{1}{c}\right)$, and for arbitrary positive constants ε and δ, consider

$$P(|\mathrm{Z}_n| > \varepsilon) = P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| \left|\frac{1}{\mathrm{Y}_n} - \frac{1}{\mathrm{c}}\right| > \varepsilon, |\mathrm{Y}_n - \mathrm{c}| < \frac{\varepsilon}{\delta}\right) +$$

$$P\left(\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right| \left|\frac{1}{\mathrm{Y}_n} - \frac{1}{\mathrm{c}}\right| > \varepsilon, |\mathrm{Y}_n - \mathrm{c}| \geq \frac{\varepsilon}{\delta}\right)$$

$$\le P\left(\frac{\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right|}{\left|\mathrm{cY}_n\right|} \ge \delta\right) + P\left(\left|\mathrm{Y}_n - \mathrm{c}\right| \ge \frac{\varepsilon}{\delta}\right).$$

For any fixed δ, taking limits of both sides of the above inequality, and noting that by assumption $\mathrm{Y}_n \xrightarrow{p} c$ and $\sum_{i=1}^{m} \mathrm{X}_{ni} \xrightarrow{d} m\mathrm{X}$, we have

$$\lim_{n\to\infty} P(|\,\mathrm{Z}_n\,| > \varepsilon) \le \lim_{n\to\infty} P\left(\frac{\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right|}{\left|\mathrm{cY}_n\right|} > \delta\right) = P\left(\frac{\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right|}{\left|\mathrm{cY}_n\right|} > \delta\right).$$

But δ is arbitrary and hence $P\left(\frac{\left|\sum_{i=1}^{m} \mathrm{X}_{ni}\right|}{\left|\mathrm{cY}_n\right|} > \delta\right)$ can be made as small as desired by choosing a large enough value for δ. Therefore $\lim_{n\to\infty} P(|\,\mathrm{Z}_n\,| > \varepsilon) = 0$ and $\mathrm{Z}_n \xrightarrow{P} 0$.

Hence by proposition (3)(*a*), $\frac{\sum_{i=1}^{m} \mathrm{X}_{ni}}{\mathrm{Y}_n}$ and $\frac{\sum_{i=1}^{m} \mathrm{X}_{ni}}{\mathrm{c}}$ will have the same asymptotic distribution given by the distribution of $\frac{m\mathrm{X}}{\mathrm{c}}$. ■

Example (7)

Suppose that $\{\mathrm{X}_n\}$ be a sequence of independent random variables and $\{\mathrm{Y}_n\}$ be a sequences of random variables such that $\mathrm{X}_{ni} \xrightarrow{d} \mathrm{X}$, i = 1, …, m and $\mathrm{Y}_n \xrightarrow{p} c$, where X~ N (μ,σ^2) and c is a constant not infinity and suppose that there are two constants such as a and b. Then

(1) $a\left(\sum_{i=1}^{m} \mathrm{X}_{ni}\right) \pm b\mathrm{Y}_n \xrightarrow{d} N(ma\mu \pm b\mathrm{c}, ma^2\sigma^2)$.

(2) $ab\left(\sum_{i=1}^{m} \mathrm{X}_{ni}\mathrm{Y}_n\right) \xrightarrow{d} N(mab\mathrm{c}\mu, ma^2b^2\mathrm{c}^2\sigma^2)$.

(3) $\frac{a\left(\sum_{i=1}^{m} \mathrm{X}_{ni}\right)}{b\mathrm{Y}_n} \xrightarrow{d} N(\frac{ma\mu}{b\mathrm{c}}, \frac{ma^2\sigma^2}{b^2\mathrm{c}^2})$.

Theorem (8)

Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of independent random variables such that $X_{ni} \xrightarrow{d} X$, $i = 1, \ldots, m$ and $Y_n \xrightarrow{d} Y$, where X and Y are two random variables. Suppose X_n and Y_n are independent for $n \geq 1$. Then X and Y are independent, and

(*a*) $\sum_{i=1}^{m} X_{ni} \pm Y_n \xrightarrow{d} mX \pm Y;$

(b) $\sum_{i=1}^{m} X_{ni} Y_n \xrightarrow{d} mXY;$

(c) $\dfrac{\sum_{i=1}^{m} X_{ni}}{Y_n} \xrightarrow{d} \dfrac{mX}{Y}.$

Proof:

(*a*) Let $\phi_n(\theta)$ be the characteristic function of $\sum_{i=1}^{m} X_{ni} + Y_n$, for every real θ.

Then

$$\phi_n(\theta) = E\left(e^{i\theta\left(\sum_{i=1}^{m} X_{ni} + Y_n\right)} \right) = E\left(e^{i\theta \sum_{i=1}^{m} X_{ni}} e^{i\theta Y_n} \right)$$

and taking limits we have:

$$\lim_{n\to\infty} \phi_n(\theta) = E\left(e^{i\theta m X} e^{i\theta Y} \right)$$

which is the characteristic function of mX+Y, and consequently

$$\sum_{i=1}^{m} X_{ni} + Y_n \xrightarrow{d} mX + Y \text{ (by Theorem (2)(i),(ii))}.$$

And to proof the other direction, we have

Let $\phi_n(\theta)$ be the characteristic function of $\sum_{i=1}^{m} X_{ni} - Y_n$, for every real θ.

Then

$$\phi_n(\theta) = E\left(e^{i\theta\left(\sum_{i=1}^{m} X_{ni} - Y_n\right)} \right) = E\left(e^{i\theta \sum_{i=1}^{m} X_{ni}} e^{-i\theta Y_n} \right)$$

and

$$\lim_{n\to\infty} \phi_n(\theta) = E\left(e^{i\theta m X} e^{-i\theta Y} \right)$$

which is the characteristic function of mX–Y, and this is implies that

$$\sum_{i=1}^{m} X_{ni} - Y_n \xrightarrow{d} mX - Y \text{ (by Theorem (2)(i),(ii))}.$$

(b) Let $\phi_n(\theta)$ be denote the characteristic function of $\sum_{i=1}^{m} X_{ni} Y_n$, for every real θ. Then

$$\phi_n(\theta) = E\left(e^{i\theta\left(\sum_{i=1}^{m} X_{ni} Y_n\right)} \right).$$

Taking limits we have:

$$\lim_{n\to\infty} \phi_n(\theta) = E\left(e^{i\theta(mXY)}\right)$$

which is the characteristic function of mXY, and consequently

$$\sum_{i=1}^{m} X_{ni} Y_n \xrightarrow{d} mXY \text{ (by Theorem (2)(i),(ii)).}$$

(c) The proof is similar to that given above for (a), (b). ■

Example (9)

Suppose that $\{X_n\}$ and $\{Y_n\}$ two sequences of independent random variables such that $X_{ni} \xrightarrow{d} X$, i = 1, …, m and $Y_n \xrightarrow{d} Y$, where X and Y are independent such that X~$N(\mu_1, \sigma_1^2)$ and Y~$N(\mu_2, \sigma_2^2)$ and suppose that there are two constants such as a and b. Then

(1) $a\left(\sum_{i=1}^{m} X_{ni}\right) \pm bY_n \xrightarrow{d} N(ma\mu_1 \pm b\mu_2, ma^2\sigma_1^2 + b^2\sigma_2^2)$.

(2) $ab\left(\sum_{i=1}^{m} X_{ni} Y_n\right) \xrightarrow{d} N(mab\mu_1\mu_2, ma^2b^2\sigma_1^2\sigma_2^2)$.

Theorem (*10*) (Velasco , 2001)

$$\underline{X}_n \xrightarrow{d} \underline{X} \Leftrightarrow \underline{\lambda}'\underline{X}_n \xrightarrow{d} \underline{\lambda}'\underline{X} \quad \forall \text{ vector } \underline{\lambda}.$$

Theorem (11) (Johnson & Wichern , 1998)

$$\underline{X} \sim N(\underline{\mu}, V) \Leftrightarrow \underline{\lambda}'\underline{X} \sim N(\underline{\lambda}'\underline{\mu}, \underline{\lambda}'V\underline{\lambda}), \forall \text{ vector } \underline{\lambda}.$$

Lemma (12) (Velasco , 2001)

Let $\{\underline{X}_n\}$ and $\{\underline{Y}_n\}$ be sequences of random ($k \times 1$) vectors. Then:

(a) If $(\underline{X}_n - \underline{Y}_n) \xrightarrow{p} \underline{0}$ and $\underline{X}_n \xrightarrow{d} \underline{X} \Rightarrow \underline{Y}_n \xrightarrow{d} \underline{X}$.

(b) If $\underline{X}_n \xrightarrow{d} \underline{X}$ and $\underline{Y}_n \xrightarrow{p} \underline{0} \Rightarrow \underline{Y}'_n \underline{X}_n \xrightarrow{p} \underline{0}$.

2. Some results for a multivariate case about the asymptotic theory .

Lemma (13)

Let $\{\underline{X}_n\}$ and $\{\underline{Z}_n\}$ be two sequences of ($k \times 1$) independent random vectors such that $Z_{1n} = X_{1n1} + X_{1n2} + ... + X_{1nm}$, $Z_{2n} = X_{2n1} + X_{2n2} + ... + X_{2nm}$ and $Z_{kn} = X_{kn1} + X_{kn2} + ... + X_{knm}$. Then $\sum_{t=1}^{m} \underline{X}_{nt} = \underline{Z}_n$

Proof

$$\sum_{t=1}^{m} \underline{X}_{nt} = \underline{X}_{n1} + \underline{X}_{n2} + ... + \underline{X}_{nm}$$

$$= \begin{pmatrix} X_{1n1} \\ X_{2n1} \\ . \\ . \\ . \\ X_{kn1} \end{pmatrix} + \begin{pmatrix} X_{1n2} \\ X_{2n2} \\ . \\ . \\ . \\ X_{kn2} \end{pmatrix} + ... + \begin{pmatrix} X_{1nm} \\ X_{2nm} \\ . \\ . \\ . \\ X_{knm} \end{pmatrix}$$

$$= \begin{pmatrix} X_{1n1} + X_{1n2} + \ . \ . \ . + X_{1nm} \\ X_{2n1} + X_{2n2} + \ . \ . \ . + X_{2nm} \\ . \qquad\qquad . \\ . \qquad\qquad . \\ . \qquad\qquad . \\ X_{kn1} + X_{kn2} + \ . \ . \ . + X_{knm} \end{pmatrix} = \begin{pmatrix} Z_{1n} \\ Z_{2n} \\ . \\ . \\ . \\ Z_{kn} \end{pmatrix} = \underline{Z}_n .$$

Lemma (14)

Let $\{\underline{X}_n\}$ be a sequence of ($k \times 1$) independent random vectors. If $\underline{X}_{nt} \xrightarrow{d} \underline{X}$; $t = 1,\dots, m$, then $\sum_{t=1}^{m} \underline{X}_{nt} \xrightarrow{d} m\underline{X}$.

Proof

Denote the characteristic functions of $\underline{\lambda}' \sum_{t=1}^{m} \underline{X}_{nt}$ and $m\underline{\lambda}'\underline{X}$ by $\phi_{\underline{\lambda}' \sum_{t=1}^{m} \underline{X}_{nt}}(\theta)$ and $\phi_{m\underline{\lambda}'\underline{X}}(\theta)$, respectively,

when $\underline{\lambda} = (\underline{\lambda}_1,...,\underline{\lambda}_k)'$ is an arbitrary vector of fixed constants and θ is any real.

By using Theorem (2)(i),(ii) and Theorem (10), we have

$$\phi_{\underline{\lambda}'\sum_{t=1}^{m}\underline{X}_{nt}}(\theta)=\phi_{\underline{\lambda}'\underline{Z}_n}(\theta) \qquad \text{(by Lemma (13))}$$

$$=E\left(e^{i\theta\underline{\lambda}'\underline{Z}_n}\right)=E\left(e^{i\theta\sum_{i=1}^{k}\lambda_i Z_{in}}\right),$$

and

$$\lim_{n\to\infty}\phi_{\underline{\lambda}'\sum_{t=1}^{m}\underline{X}_{nt}}(\theta)=E\left(e^{i\theta\sum_{i=1}^{k}\lambda_i Z_i}\right)=E\left(e^{i\theta m\sum_{i=1}^{k}\lambda_i X_i}\right)$$

$$=E\left(e^{i\theta m\underline{\lambda}'\underline{X}}\right)=\phi_{m\underline{\lambda}'\underline{X}}(\theta)$$

$$\lim_{n\to\infty}\phi_{\underline{\lambda}'\sum_{t=1}^{m}\underline{X}_{nt}}(\theta)=\phi_{m\underline{\lambda}'\underline{X}}(\theta),\forall\underline{\lambda}\in R^k,\underline{\lambda}\neq\underline{0},\forall\theta\in R$$

$$\Rightarrow\underline{\lambda}'\sum_{t=1}^{m}\underline{X}_{nt}\xrightarrow{d}m\underline{\lambda}'\underline{X},\forall\underline{\lambda}\in R^k,\underline{\lambda}\neq\underline{0} \quad \text{(by Theorem (2)(i),(ii))}$$

$$\Rightarrow\sum_{t=1}^{m}\underline{X}_{nt}\xrightarrow{d}m\underline{X} \quad \text{(by Theorem (10))}.$$

Theorem (15)

Let $\{\underline{X}_n\}$ be a sequence of (k × 1) independent random vectors with $\underline{X}_{nt}\xrightarrow{d}\underline{X}$, t = 1,…, m, and let $\{\underline{Y}_n\}$ be a sequence of (k × 1) random vectors with $\underline{Y}_n\xrightarrow{p}\underline{c}$, where $\underline{c}$ be a vector of constants not infinity. Then

(a) $\sum_{t=1}^{m}\underline{X}_{nt}\pm\underline{Y}_n\xrightarrow{d}m\underline{X}\pm\underline{c}$;

(b) $\underline{Y}'_n\sum_{t=1}^{m}\underline{X}_{nt}\xrightarrow{d}m\underline{c}'\underline{X}$.

Proof

(a) Choose and fix v such that $v-\sum_{i=1}^{k}\lambda_i c_i$ is a continuity point of $F_{m\sum_{i=1}^{k}\lambda_i X_i}(v)$. Let $\varepsilon>0$ be such that $v-\sum_{t=1}^{k}\lambda_i c_i+\varepsilon$ and $v-\sum_{i=1}^{k}\lambda_i c_i-\varepsilon$ are also continuity points of $F_{m\sum_{i=1}^{k}\lambda_i X_i}(v)$.

Denote the distribution functions of $\underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt}+\underline{Y}_n\right)$ and $\underline{\lambda}'(m\underline{X}+\underline{c})$ by $F_{\underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt}+\underline{Y}_n\right)}(v)$ and $F_{\underline{\lambda}'(m\underline{X}+\underline{c})}(v)$, respectively,

when $\underline{\lambda}=(\lambda_1,\ldots,\lambda_k)'$ is an arbitrary vector of fixed constants. By using Definition (1) and Theorem (10), we have

$$F_{\underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt}+\underline{Y}_n\right)}(v)=F_{\underline{\lambda}'(\underline{Z}_n+\underline{Y}_n)}(v)$$

$$=F_{\sum_{i=1}^{k}\lambda_i Z_{in}+\sum_{i=1}^{k}\lambda_i Y_{in}}(v)$$

$$=P(\sum_{i=1}^{k}\lambda_i Z_{in}+\sum_{i=1}^{k}\lambda_i Y_{in}\le v)$$

$$\le P\left(\sum_{i=1}^{k}\lambda_i Z_{in}+\sum_{i=1}^{k}\lambda_i Y_{in}\le v,\left|\sum_{i=1}^{k}\lambda_i Y_{in}-\sum_{i=1}^{k}\lambda_i c_i\right|<\varepsilon\right)$$
$$+P\left(\left|\sum_{i=1}^{k}\lambda_i Y_{in}-\sum_{i=1}^{k}\lambda_i c_i\right|\ge\varepsilon\right)$$

$$\le P\left(\sum_{i=1}^{k}\lambda_i Z_{in}\le v-\sum_{i=1}^{k}\lambda_i c_i+\varepsilon\right)+P\left(\left|\sum_{i=1}^{k}\lambda_i Y_{in}-\sum_{i=1}^{k}\lambda_i c_i\right|\ge\varepsilon\right).$$

Hence, by the hypotheses of the theorem, and by the choice of $v-\sum_{i=1}^{k}\lambda_i c_i+\varepsilon$,

$$\limsup_{n\to\infty}F_{\sum_{i=1}^{k}\lambda_i Z_{in}+\sum_{i=1}^{k}\lambda_i Y_{in}}(v)$$

$$\le\limsup_{n\to\infty}P\left(\sum_{i=1}^{k}\lambda_i Z_{in}\le v-\sum_{i=1}^{k}\lambda_i c_i+\varepsilon\right)+\limsup_{n\to\infty}P\left(\left|\sum_{i=1}^{k}\lambda_i Y_{in}-\sum_{i=1}^{k}\lambda_i c_i\right|\ge\varepsilon\right)$$
$$=F_{\sum_{i=1}^{k}\lambda_i Z_i}\left(v-\sum_{i=1}^{k}\lambda_i c_i+\varepsilon\right)=F_{m\sum_{i=1}^{k}\lambda_i X_i}\left(v-\sum_{i=1}^{k}\lambda_i c_i+\varepsilon\right).$$

Similarly,

$$P\left(\sum_{i=1}^{k}\lambda_i Z_{in} \le v - \sum_{i=1}^{k}\lambda_i c_i - \varepsilon\right) \le P\left(\sum_{i=1}^{k}\lambda_i Z_{in} + \sum_{i=1}^{k}\lambda_i Y_{in} \le v\right) + P\left(\left|\sum_{i=1}^{k}\lambda_i Y_{in} - \sum_{i=1}^{k}\lambda_i c_i\right| \ge \varepsilon\right)$$

and thus

$$F_{\sum_{i=1}^{k}\lambda_i Z_i}\left(v - \sum_{i=1}^{k}\lambda_i c_i - \varepsilon\right) = F_{m\sum_{i=1}^{k}\lambda_i X_i}\left(v - \sum_{i=1}^{k}\lambda_i c_i - \varepsilon\right) \le \liminf_{n\to\infty} F_{\sum_{i=1}^{k}\lambda_i Z_{in} + \sum_{i=1}^{k}\lambda_i Y_{in}}(v).$$

Since $v - \sum_{i=1}^{k}\lambda_i c_i$ is a continuity point of $F_{m\sum_{i=1}^{k}\lambda_i X_i}(v)$, and since ε may be taken arbitrarily small, we have

$$\lim_{n\to\infty} F_{\sum_{i=1}^{k}\lambda_i Z_{in} + \sum_{i=1}^{k}\lambda_i Y_{in}}(v) = F_{\sum_{i=1}^{k}\lambda_i Z_i}\left(v - \sum_{i=1}^{k}\lambda_i c_i\right) = F_{m\sum_{i=1}^{k}\lambda_i X_i}\left(v - \sum_{i=1}^{k}\lambda_i c_i\right)$$

$$= F_{m\sum_{i=1}^{k}\lambda_i X_i + \sum_{i=1}^{k}\lambda_i c_i}(v)$$

i.e., $$\lim_{n\to\infty} F_{\underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt} + \underline{Y}_n\right)}(v) = F_{\underline{\lambda}'(m\underline{X}+\underline{c})}(v)$$

$$\Rightarrow \underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt} + \underline{Y}_n\right) \xrightarrow{d} \underline{\lambda}'(m\underline{X} + \underline{c}), \forall \underline{\lambda} \in R^k, \underline{\lambda} \ne \underline{0}$$

(by Definition (1))

$$\Rightarrow \sum_{t=1}^{m}\underline{X}_{nt} + \underline{Y}_n \xrightarrow{d} m\underline{X} + \underline{c} \text{ (by Theorem (10)).}$$

The proof of the other direction is similar to that given above.

(b) Denote the characteristic functions of $\underline{Y}'_n \sum_{t=1}^{m}\underline{X}_{nt}$ and $m\underline{c}'\underline{X}$ by $\phi_{\underline{Y}'_n \sum_{t=1}^{m}\underline{X}_{nt}}(\theta)$ and $\phi_{m\underline{c}'\underline{X}}(\theta)$, respectively, when θ is any real.

By using Theorem (2)(i),(ii), we have

$$\phi_{\underline{Y}'_n \sum_{t=1}^{m}\underline{X}_{nt}}(\theta) = \phi_{\underline{Y}'_n \underline{Z}_n}(\theta) \text{ (by Lemma (13))}$$

$$= E\left(e^{i\theta \underline{Y}'_n \underline{Z}_n}\right) = E\left(e^{i\theta\sum_{i=1}^{k} Y_{in} Z_{in}}\right),$$

and hence

$$\lim_{n\to\infty}\phi_{\underline{Y}'_n \sum_{t=1}^{m}\underline{X}_{nt}}(\theta) = E\left(e^{i\theta\sum_{i=1}^{k} c_i Z_i}\right) = E\left(e^{i\theta m\sum_{i=1}^{k} c_i X_i}\right)$$

$$= E\left(e^{i\theta m\underline{c}'\underline{X}}\right) = \phi_{m\underline{c}'\underline{X}}(\theta)$$

$$\lim_{n\to\infty} \phi_{\underline{Y}'_n \sum_{t=1}^{m} \underline{X}_{nt}}(\theta) = \phi_{m\underline{c}'\underline{X}}(\theta), \forall\theta \in R$$, it then follows that

$$\underline{Y}'_n \sum_{t=1}^{m} \underline{X}_{nt} \xrightarrow{d} m\underline{c}'\underline{X}$$ (by Theorem (2)(i),(ii)). ■

Example (16)

Let $\{\underline{X}_n\}$ be a sequence of $(k \times 1)$ independent random vectors with $\underline{X}_{nt} \xrightarrow{d} \underline{X}$, $t = 1,\dots, m$, where $\underline{X} \sim N(\underline{\mu}, V)$ and let $\{\underline{Y}_n\}$ be a sequence of $(k\times 1)$ random vectors with $\underline{Y}_n \xrightarrow{p} \underline{c}$, where $\underline{c}$ be a vector of constants not infinity. Then

(1) The limiting distribution of $\sum_{t=1}^{m} \underline{X}_{nt} \pm \underline{Y}_n$ is the same as that of $m\underline{X} \pm \underline{c}$; that is,

using Theorem (11), we obtain that

$$\forall \underline{\lambda} = (\lambda_1,\dots,\lambda_k)' \in R^k, \underline{\lambda} \neq \underline{0}$$

$$\underline{\lambda}'(m\underline{X}) \sim N(m\underline{\lambda}'\underline{\mu}, m\underline{\lambda}'V\underline{\lambda}).$$

Using the other direction of the Theorem (11), we now find

$$m\underline{X} \sim N(m\underline{\mu}, mV)$$

where mV is positive definite because $0 < m\underline{\lambda}'V\underline{\lambda} < \infty$ for all $\underline{\lambda} \neq \underline{0}$. Finally, it follows that

$$m\underline{X} \pm \underline{c} \sim N(m\underline{\mu} \pm \underline{c}, mV)$$ such that

$$\sum_{t=1}^{m} \underline{X}_{nt} \pm \underline{Y}_n \xrightarrow{d} N(m\underline{\mu} \pm \underline{c}, mV).$$

(2) The limiting distribution of $\underline{Y}'_n \sum_{t=1}^{m} \underline{X}_{nt}$ is the same as that of $m\underline{c}'\underline{X}$. Note that

$$m\underline{c}'\underline{X} \sim N(m\underline{c}'\underline{\mu}, m\underline{c}'V\underline{c})$$, and this is follows that

$$\underline{Y}'_n \sum_{t=1}^{m} \underline{X}_{nt} \xrightarrow{d} N(m\underline{c}'\underline{\mu}, m\underline{c}'V\underline{c}).$$

Theorem (17)

Let $\{\underline{X}_n\}$ be a sequence of ($k \times 1$) independent random vectors with $\underline{X}_{nt} \xrightarrow{d} \underline{X}$, $t = 1,\ldots, m$, and let $\{\underline{Y}_n\}$ be a sequence of ($k \times 1$) independent random vectors with $\underline{Y}_n \xrightarrow{d} \underline{Y}$, where $\underline{X}$ and $\underline{Y}$ are two random vectors. Suppose $\underline{X}_n$ and $\underline{Y}_n$ are independent for $n \geq 1$. Then $\underline{X}$ and $\underline{Y}$ are independent, and

(a) $\sum_{t=1}^{m} \underline{X}_{nt} \pm \underline{Y}_n \xrightarrow{d} m\underline{X} \pm \underline{Y}$;

(b) $\underline{Y}'_n \sum_{t=1}^{m} \underline{X}_{nt} \xrightarrow{d} m\underline{Y}'\underline{X}$.

Proof:

(a) Let $\phi_{\underline{\lambda}'\left(\sum_{t=1}^{m} \underline{X}_{nt} + \underline{Y}_n\right)}(\theta)$ be the characteristic function of $\underline{\lambda}'\left(\sum_{t=1}^{m} \underline{X}_{nt} + \underline{Y}_n\right)$ and let $\phi_{\underline{\lambda}'(m\underline{X}+\underline{Y})}(\theta)$ be the characteristic function of $\underline{\lambda}'(m\underline{X} + \underline{Y})$, for any vector $\underline{\lambda} = (\lambda_1,\ldots,\lambda_k)' \in R^k$ and any real θ.

Using for proof Theorem (2)(i),(ii) and Theorem (10), we have

$$\phi_{\underline{\lambda}'\left(\sum_{t=1}^{m} \underline{X}_{nt} + \underline{Y}_n\right)}(\theta) = \phi_{\underline{\lambda}'(\underline{Z}_n + \underline{Y}_n)}(\theta) \qquad \text{(by Lemma (13))}$$

$$= E\left(e^{i\theta\underline{\lambda}'(\underline{Z}_n + \underline{Y}_n)}\right)$$

$$= E\left(e^{i\theta\left(\sum_{i=1}^{k} \lambda_i Z_{in} + \sum_{i=1}^{k} \lambda_i Y_{in}\right)}\right)$$

$$= E\left(e^{i\theta\sum_{i=1}^{k} \lambda_i Z_{in}}\, e^{i\theta\sum_{i=1}^{k} \lambda_i Y_{in}}\right).$$

Now taking limits of the above, we have

$$\lim_{n\to\infty} \phi_{\underline{\lambda}'\left(\sum_{t=1}^{m} \underline{X}_{nt} + \underline{Y}_n\right)}(\theta) = E\left(e^{i\theta\sum_{i=1}^{k} \lambda_i Z_i}\, e^{i\theta\sum_{i=1}^{k} \lambda_i Y_i}\right) = E\left(e^{i\theta m\sum_{i=1}^{k} \lambda_i X_i}\, e^{i\theta\sum_{i=1}^{k} \lambda_i Y_i}\right)$$

$$= E\left(e^{i\theta\left(m\sum_{i=1}^{k} \lambda_i X_i + \sum_{i=1}^{k} \lambda_i Y_i\right)}\right)$$

$$= E\left(e^{i\theta\underline{\lambda}'(m\underline{X}+\underline{Y})}\right) = \phi_{\underline{\lambda}'(m\underline{X}+\underline{Y})}(\theta).$$

Therefore, we have

$$\underline{\lambda}'\left(\sum_{t=1}^{m}\underline{X}_{nt}+\underline{Y}_n\right)\xrightarrow{d}\underline{\lambda}'(m\underline{X}+\underline{Y}),\forall\underline{\lambda}\in R^k,\underline{\lambda}\neq\underline{0}$$ (by Theorem (2)(i),(ii)). It then follows that

$$\sum_{t=1}^{m}\underline{X}_{nt}+\underline{Y}_n\xrightarrow{d}m\underline{X}+\underline{Y}$$ (by Theorem (10)).

The proof of the other direction is similar to that given above. ■
(b) the proof is similar to that given above for (*a*).

Example (18)

Let $\{\underline{X}_n\}$ be denote a sequence of (k × 1) independent random vectors with $\underline{X}_{nt}\xrightarrow{d}\underline{X}$, t = 1,…, m, and let $\{\underline{Y}_n\}$ be denote a sequence of (k × 1) independent random vectors with $\underline{Y}_n\xrightarrow{d}\underline{Y}$. Suppose that $\underline{X}_n$ and $\underline{Y}_n$ are independent for $n\geq 1$. Then $\underline{X}$ and $\underline{Y}$ are independent such that $\underline{X}\sim N(\underline{\mu}_1,V_1)$ and $\underline{Y}\sim N(\underline{\mu}_2,V_2)$, and the limiting distribution of $\sum_{t=1}^{m}\underline{X}_{nt}\pm\underline{Y}_n$ is the same as that of $m\underline{X}\pm\underline{Y}$; that is, note that by taking Theorem (11), we obtain that

$$\forall\underline{\lambda}=(\lambda_1,\ldots,\lambda_k)'\in R^{\mathrm{k}},\underline{\lambda}\neq\underline{0}$$

$$\underline{\lambda}'(m\underline{X}\pm\underline{Y})\sim N(m\underline{\lambda}'\underline{\mu}_1\pm\underline{\lambda}'\underline{\mu}_2,m\underline{\lambda}'V_1\underline{\lambda}+\underline{\lambda}'V_2\underline{\lambda}).$$

Using the other direction of the Theorem (11), we now find

$$m\underline{X}\pm\underline{Y}\sim N(m\underline{\mu}_1\pm\underline{\mu}_2,mV_1+V_2)$$

where mV_1+V_2 is positive definite since $\forall\underline{\lambda}\in R^k,\underline{\lambda}\neq\underline{0}$: $m\underline{\lambda}'V_1\underline{\lambda}+\underline{\lambda}'V_2\underline{\lambda}>0$, it then follows that

$$\sum_{t=1}^{m}\underline{X}_{nt}\pm\underline{Y}_n\xrightarrow{d}N(m\underline{\mu}_1\pm\underline{\mu}_2,mV_1+V_2).$$

Theorem (19)

Let $\{\underline{X}_n\}$ be a sequence of (k × 1) independent random vectors with $\underline{X}_{nt}\xrightarrow{d}\underline{X}$, t = 1,…, m, and let $\{\mathrm{Y}_n\}$ be a sequence of (ω × k) random matrices with $\mathrm{Y}_n\xrightarrow{p}\mathrm{C}$. Then $\mathrm{Y}_n\sum_{t=1}^{m}\underline{X}_{nt}\xrightarrow{d}m\mathrm{C}\underline{X}$.

Proof:

Let $\phi_{\underline{\lambda}'\left(\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}\right)}(\theta)$ and $\phi_{\underline{\lambda}'(m\mathrm{C}\underline{\mathrm{X}})}(\theta)$ be the characteristic functions of $\underline{\lambda}'\left(\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}\right)$ and $\underline{\lambda}'(m\mathrm{C}\underline{\mathrm{X}})$, respectively, when $\underline{\lambda} = (\lambda_1,\ldots,\lambda_\omega)'$ is an arbitrary vector of fixed constants and θ is any real.

$$\phi_{\underline{\lambda}'\left(\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}\right)}(\theta) = \phi_{\underline{\lambda}'(\mathrm{Y}_n \underline{\mathrm{Z}}_n)}(\theta) \qquad \text{(by Lemma (13))}$$

$$= E\left(\mathrm{e}^{i\theta\underline{\lambda}'(\mathrm{Y}_n \underline{\mathrm{Z}}_n)}\right) = E\left(\mathrm{e}^{\mathrm{i}\theta \sum_{j=1}^{\omega} \sum_{i=1}^{k} \lambda_j \mathrm{y}_{jin} \mathrm{Z}_{in}}\right),$$

and hence

$$\lim_{n\to\infty} \phi_{\underline{\lambda}'\left(\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}\right)}(\theta) = E\left(\mathrm{e}^{i\theta \sum_{j=1}^{\omega} \sum_{i=1}^{k} \lambda_j \mathrm{c}_{ji} Z_i}\right) = E\left(\mathrm{e}^{i\theta m \sum_{j=1}^{\omega} \sum_{i=1}^{k} \lambda_j \mathrm{c}_{ji} \mathrm{X}_i}\right)$$

$$= E\left(\mathrm{e}^{i\theta\underline{\lambda}'(m\mathrm{C}\underline{\mathrm{X}})}\right) = \phi_{\underline{\lambda}'(m\mathrm{C}\underline{\mathrm{X}})}(\theta).$$

$$\Rightarrow \underline{\lambda}'\left(\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}\right) \xrightarrow{d} \underline{\lambda}'(m\mathrm{C}\underline{\mathrm{X}}), \forall \underline{\lambda} \in R^{\omega}, \underline{\lambda} \neq \underline{0} \text{ (by Theorem(2)(i),(ii))}.$$

$$\Rightarrow \mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt} \xrightarrow{d} m\mathrm{C}\underline{\mathrm{X}} \text{ (by Theorem (10)).} \blacksquare$$

Example (20)

Let $\{\underline{\mathrm{X}}_n\}$ be denote a sequence of ($k \times 1$) independent random vectors with $\underline{\mathrm{X}}_{nt} \xrightarrow{d} \underline{\mathrm{X}}$, $t = 1,\ldots, m$, where $\underline{\mathrm{X}} \sim N(\underline{\mu}, V)$ and let $\{\mathrm{Y}_n\}$ be denote a sequence of ($\omega \times k$) random matrices with $\mathrm{Y}_n \xrightarrow{p} \mathrm{C}$. Then the limiting distribution of $\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt}$ is the same as that of $m\mathrm{C}\underline{\mathrm{X}}$, in other words,

$$\mathrm{Y}_n \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt} \xrightarrow{d} N(m\mathrm{C}\underline{\mu}, m\mathrm{C}V\mathrm{C}').$$

Theorem (21)

Let $\{\underline{\mathrm{X}}_n\}$ be a sequence of ($k \times 1$) independent random vectors with $\underline{\mathrm{X}}_{nt} \xrightarrow{d} \underline{\mathrm{X}}$, $t = 1,\ldots, m$ and let $\{\mathrm{Y}_n\}$ be a sequence of ($k \times k$) random matrices with $\mathrm{Y}_n \xrightarrow{p} \mathrm{C}$, a nonsingular matrix. Then $\mathrm{Y}_n^{-1} \sum_{t=1}^{m} \underline{\mathrm{X}}_{nt} \xrightarrow{d} m\mathrm{C}^{-1}\underline{\mathrm{X}}$.

Proof

Suppose that

$$\mathrm{Y}_n^{-1} = \begin{bmatrix} a_{11n} & . & . & . & a_{1kn} \\ . & & & & \\ . & & & & \\ . & & & & \\ a_{k1n} & . & . & . & a_{kkn} \end{bmatrix} \text{ and } \mathrm{C}^{-1} = \begin{bmatrix} d_{11} & . & . & . & d_{1k} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ d_{k1} & . & . & . & d_{kk} \end{bmatrix}.$$

Now using for proof Theorem (2)(i),(ii) and Theorem (10) by denoting the characteristic functions of $\underline{\lambda}'\left(\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\right)$ and $\underline{\lambda}'(m\mathrm{C}^{-1}\underline{\mathrm{X}})$ by

$\phi_{\underline{\lambda}'\left(\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\right)}(\theta)$ and $\phi_{\underline{\lambda}'(m\mathrm{C}^{-1}\underline{\mathrm{X}})}(\theta)$, respectively, $\forall\underline{\lambda}=(\lambda_1,\ldots,\lambda_k)'\in R^k$, $\underline{\lambda}\neq\underline{0},\forall\theta\in R$:

$$\phi_{\underline{\lambda}'\left(\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\right)}(\theta)=\phi_{\underline{\lambda}'(\mathrm{Y}_n^{-1}\underline{\mathrm{Z}}_n)}(\theta) \qquad \text{(by Lemma (13))}$$

$$=E\left(\mathrm{e}^{i\theta\underline{\lambda}'(\mathrm{Y}_n^{-1}\underline{\mathrm{Z}}_n)}\right)=E\left(\mathrm{e}^{\mathrm{i}\theta\sum_{j=1}^{k}\sum_{i=1}^{k}\lambda_j a_{jin}\mathrm{Z}_{in}}\right),$$

by taking limits, we obtain that

$$\lim_{n\to\infty}\phi_{\underline{\lambda}'\left(\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\right)}(\theta)=E\left(\mathrm{e}^{i\theta\sum_{j=1}^{k}\sum_{i=1}^{k}\lambda_j d_{ji}Z_i}\right)=E\left(\mathrm{e}^{i\theta m\sum_{j=1}^{k}\sum_{i=1}^{k}\lambda_j d_{ji}\mathrm{X}_i}\right)$$

$$=E\left(\mathrm{e}^{i\theta\underline{\lambda}'(m\mathrm{C}^{-1}\underline{\mathrm{X}})}\right)=\phi_{\underline{\lambda}'(m\mathrm{C}^{-1}\underline{\mathrm{X}})}(\theta).$$

Hence, by above, we have

$$\underline{\lambda}'\left(\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\right)\xrightarrow{d}\underline{\lambda}'(m\mathrm{C}^{-1}\underline{\mathrm{X}}),\forall\underline{\lambda}\in R^k,\underline{\lambda}\neq\underline{0} \qquad \text{(by Theorem}$$

(2)(i),(ii)), and this is follows that

$$\mathrm{Y}_n^{-1}\sum_{t=1}^{m}\underline{\mathrm{X}}_{nt}\xrightarrow{d}m\mathrm{C}^{-1}\underline{\mathrm{X}} \text{ (by Theorem (10)).} \blacksquare$$

Example (22)

Let $\{\underline{\mathrm{X}}_n\}$ be denote a sequence of (k × 1) independent random vectors with $\underline{\mathrm{X}}_{nt}\xrightarrow{d}\underline{\mathrm{X}}$, t = 1,…, m where $\underline{\mathrm{X}}\sim N(\underline{\mu},V)$ and let $\{\mathrm{Y}_n\}$ be denote a sequence of (k

× k) random matrices with $Y_n \xrightarrow{p} C$. Then the limiting distribution of $Y_n^{-1}\sum_{t=1}^{m}\underline{X}_{nt}$ is the same as that of $mC^{-1}\underline{X}$, in other words

$$Y_n^{-1}\sum_{t=1}^{m}\underline{X}_{nt} \xrightarrow{d} N\left(mC^{-1}\underline{\mu}, mC^{-1}V[C^{-1}]'\right).$$

Lemma (23)

let $\{\underline{X}_n\}$ be a sequence of (k ×1) random vectors. If $\underline{X}_{nt} \xrightarrow{p} \underline{c}$, t = 1, …, m. Then $\sum_{t=1}^{m}\underline{X}_{nt} \xrightarrow{p} m\underline{c}$.

Theorem (24)

Let $\{\underline{X}_n\}$ be a sequence of (k ×1) random vectors with $\underline{X}_{nt} \xrightarrow{p} \underline{c}_1$, t = 1,…, m and let $\{Y_n\}$ be a sequence of (k × k) random matrices with $Y_n \xrightarrow{p} C_2$, a nonsingular matrix. Then $Y_n^{-1}\sum_{t=1}^{m}\underline{X}_{nt} \xrightarrow{p} mC_2^{-1}\underline{c}_1$.

Proof

To proof this, note that the elements of the matrix Y_n^{-1} are continuous functions of the elements of Y_n at $Y_n = C_2$, since C_2^{-1} exists. Thus, $Y_n^{-1} \to C_2^{-1}$. Similarly, the elements of $Y_n^{-1}\sum_{t=1}^{m}\underline{X}_{nt}$ are sums of products of elements of Y_n^{-1} with those of $\sum_{t=1}^{m}\underline{X}_{nt}$. Since each sum is again a continuous function of Y_n^{-1} and $\sum_{t=1}^{m}\underline{X}_{nt}$,

$$\operatorname*{plim}_{n\to\infty}\left(Y_n^{-1}\sum_{t=1}^{m}\underline{X}_{nt}\right) = \left(\operatorname*{plim}_{n\to\infty} Y_n\right)^{-1}\operatorname*{plim}_{n\to\infty}\sum_{t=1}^{m}\underline{X}_{nt} = mC_2^{-1}\underline{c}_1 . \blacksquare$$

Example (25)

Let $\{\underline{X}_n\}$ be denote a sequence of (k × 1) random vectors with $\underline{X}_{nt} \xrightarrow{p} \underline{c}_1$, t = 1,…, m and let $\{\underline{Y}_n\}$ be denote a sequence of (k × 1) random vectors with $\underline{Y}_n \xrightarrow{p} \underline{c}_2$ where $\underline{c}_1$ and $\underline{c}_2$ are two vectors of constants. Then

(1) $\operatorname*{plim}_{n\to\infty}\left(\sum_{t=1}^{m}\underline{X}_{nt} \pm \underline{Y}_n\right) = \operatorname*{plim}_{n\to\infty}\sum_{t=1}^{m}\underline{X}_{nt} \pm \operatorname*{plim}_{n\to\infty}\underline{Y}_n = m\underline{c}_1 \pm \underline{c}_2$.

(2) $\underset{n\to\infty}{\text{plim}}\left(\underline{\text{Y}}_n'\sum_{t=1}^{m}\underline{\text{X}}_{nt}\right)=\left(\underset{n\to\infty}{\text{plim}}\,\underline{\text{Y}}_n\right)'\underset{n\to\infty}{\text{plim}}\sum_{t=1}^{m}\underline{\text{X}}_{nt}=m\underline{\text{c}}'_2\underline{\text{c}}_1.$

Theorem (26)

Suppose that $\underline{\text{X}}_n=(\text{X}_{1n},\ldots,\text{X}_{kn})'$ is asymptotically distributed as $N(\underline{\mu}_1,V_1/n)$ and $\underline{\text{Y}}_n=(\text{Y}_{1n},\ldots,\text{Y}_{kn})'$ is asymptotically distributed as $N(\underline{\mu}_2,V_2/n)$, where V_1 and V_2 are two fixed matrices, $\underline{\text{X}}_n$ and $\underline{\text{Y}}_n$ are independent. Consider the two random vectors $\underline{\text{X}}=(\text{X}_1,\ldots,\text{X}_k)'$ and $\underline{\text{Y}}=(\text{Y}_1,\ldots,\text{Y}_k)'$, let $g(\underline{\text{X}})=[g_1(\underline{\text{X}}),\ldots,g_\ell(\underline{\text{X}})]'$ be a vector-valued function with non-zero differentials

$$\text{G}=\left[\frac{\partial g_{\text{j}}(\underline{\text{X}})}{\partial \text{X}_i}\right]\Bigg|_{\underline{\text{X}}=\underline{\mu}_1}$$

which is an $\ell\times k$ matrix, let $f(\underline{\text{Y}})=[f_1(\underline{\text{Y}}),\ldots,f_\ell(\underline{\text{Y}})]'$ be a vector-valued function with non-zero differentials

$$\text{F}=\left[\frac{\partial f_{\text{j}}(\underline{\text{Y}})}{\partial \text{Y}_i}\right]\Bigg|_{\underline{\text{Y}}=\underline{\mu}_2}$$

which is an $\ell\times k$ matrix. Suppose $g(\underline{\text{X}}_n)$ and $f(\underline{\text{Y}}_n)$ are independent. Then the asymptotic distribution of $g(\underline{\text{X}}_n)+f(\underline{\text{Y}}_n)$ also normal with mean $g(\underline{\mu}_1)+f(\underline{\mu}_2)$ and covariance matrix $(\text{G}V_1\text{G}'+\text{F}V_2\text{F}')/n$.

Proof

Since $\text{Var}(\underline{\text{X}}_n)=\text{V}_1/n\to 0$ as $n\to\infty$, it then follows that $\underline{\text{X}}_n\xrightarrow{p}\underline{\mu}_1$, and

$$\|\underline{\text{X}}_n-\underline{\mu}_1\|=o_p(1),$$

and since $\text{Var}(\underline{\text{Y}}_n)=V_2/n\to 0$, as $n\to\infty$, it then follows that $\underline{\text{Y}}_n\xrightarrow{p}\underline{\mu}_2$, and

$$\|\underline{\text{Y}}_n-\underline{\mu}_2\|=o_p(1).$$

Now using the Taylor series, approximation result for stochastic processes we have:

$$g(\underline{\text{X}}_n)=g(\underline{\mu}_1)+\text{G}(\underline{\text{X}}_n-\underline{\mu}_1)+\text{Z}_n$$

where $\text{Z}_n=o_p(\|\underline{\text{X}}_n-\underline{\mu}_1\|$, and

$$f(\underline{\text{Y}}_n)=f(\underline{\mu}_2)+\text{F}(\underline{\text{Y}}_n-\underline{\mu}_2)+\text{H}_n$$

where $\text{H}_n=o_p(\|\underline{\text{Y}}_n-\underline{\mu}_2\|$, and this is follows that

$$g(\underline{\text{X}}_n)+f(\underline{\text{Y}}_n)=g(\underline{\mu}_1)+f(\underline{\mu}_2)+\text{G}(\underline{\text{X}}_n-\underline{\mu}_1)+\text{F}(\underline{\text{Y}}_n-\underline{\mu}_2)+\text{Z}_n+\text{H}_n.$$

Now using Lemma (12) (*a*), $\sqrt{n}\left\{g(\underline{X}_n)+f(\underline{Y}_n)-g(\underline{\mu}_1)-f(\underline{\mu}_2)\right\}$ and $\sqrt{n}\left\{G(\underline{X}_n-\underline{\mu}_1)+F(\underline{Y}_n-\underline{\mu}_2)\right\}$, will have the same limiting distribution if we show that

$$\operatorname{plim}_{n\to\infty}\left\{\sqrt{n}(Z_n+H_n)\right\}=\underline{0}.$$

But since $Z_n=o_p(\|\underline{X}_n-\underline{\mu}_1\|=o_p(1)$ and $H_n=o_p(\|\underline{Y}_n-\underline{\mu}_2\|=o_p(1)$, then by Definition (4)

$$\operatorname*{plim}_{n\to\infty}\left\{\frac{Z_n}{\|\underline{X}_n-\underline{\mu}_1\|}\right\}=\underline{0} \qquad \text{or} \qquad \operatorname*{plim}_{n\to\infty}\left\{\frac{\sqrt{n}Z_n}{\sqrt{n}\|\underline{X}_n-\underline{\mu}_1\|}\right\}=\underline{0}.$$

and

$$\operatorname*{plim}_{n\to\infty}\left\{\frac{H_n}{\|\underline{Y}_n-\underline{\mu}_2\|}\right\}=\underline{0} \qquad \text{or} \qquad \operatorname*{plim}_{n\to\infty}\left\{\frac{\sqrt{n}H_n}{\sqrt{n}\|\underline{Y}_n-\underline{\mu}_2\|}\right\}=\underline{0}.$$

However, by assumption $\sqrt{n}(\underline{X}_n-\underline{\mu})$ has a finite limiting normal distribution and is bounded stochastically,
that is

$$\sqrt{n}\|\underline{X}_n-\underline{\mu}_1\|=o_p(1),$$

and also $\sqrt{n}(\underline{Y}_n-\underline{\mu}_2)$ has a finite limiting normal distribution and bounded stochastically,
that is

$$\sqrt{n}\|\underline{Y}_n-\underline{\mu}_2\|=o_p(1)$$

and therefore we have
$\operatorname{plim}_{n\to\infty}\sqrt{n}(Z_n)=\underline{0}$ and $\operatorname{plim}_{n\to\infty}\sqrt{n}(H_n)=\underline{0}$, and this is follows that

$$\operatorname{plim}_{n\to\infty}\left\{\sqrt{n}(Z_n+H_n)\right\}=\underline{0}.$$

Hence

$$\begin{aligned}&\sqrt{n}\left\{g(\underline{X}_n)+f(\underline{Y}_n)-g(\underline{\mu}_1)-f(\underline{\mu}_2)\right\}\\&=\sqrt{n}\left\{g(\underline{X}_n)-g(\underline{\mu}_1)\right\}+\sqrt{n}\left\{f(\underline{Y}_n)-f(\underline{\mu}_2)\right\}\\&\xrightarrow{d}G\left\{\sqrt{n}(\underline{X}_n-\underline{\mu}_1)\right\}+F\left\{\sqrt{n}(\underline{Y}_n-\underline{\mu}_2)\right\}\xrightarrow{d}N(\underline{0},GV_1G'+FV_2F').\end{aligned}$$

Theorem (27)

Suppose that $\underline{X}_n=(X_{1n},\ldots,X_{kn})'$ is asymptotically distributed as $N(\underline{\mu},V/n)$, where V is a fixed matrix. Consider the random vector $\underline{X}=(X_1,\ldots,X_k)'$, let $g(\underline{X})=[g_1(\underline{X}),\ldots,g_\ell(\underline{X})]'$ be a vector-valued function with non-zero differentials

$$G=\left[\frac{\partial g_j(\underline{X})}{\partial X_i}\right]\Bigg|_{\underline{X}=\underline{\mu}}$$

which is an $\ell \times k$ matrix, let $f(\underline{X}) = [f_1(\underline{X}),\ldots, f_\ell(\underline{X})]'$ be a vector-valued function with non-zero differentials

$$F = \left[\frac{\partial f_j(\underline{X})}{\partial X_i}\right]\Bigg|_{\underline{X}=\underline{\mu}}$$

which is an $\ell \times k$ matrix. Suppose $g(\underline{X}_n)$ and $f(\underline{X}_n)$ are independent. Then the asymptotic distribution of $g(\underline{X}_n) + f(\underline{X}_n)$ also normal with mean $g(\underline{\mu}) + f(\underline{\mu})$ and covariance matrix $(GVG' + FVF')/n$.

Proof

The proof is the same way for proof theorem (26) .

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