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The Dynamics of the Fixed Points to Modified Jerk Map

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Abstract: Recently, Jerk equation is the third-order explicit autonomous differential equation, is noticed to be a motivating sub-class of dynamical systems. Where many of regular and chaotic motion features can be reveal from these systems. In this paper, a simplified version Jerk map is presented. Different properties of dynamical behavior is acquired by replacing three dimensional systems to two dimensional one. Where a new parameter is added with the same properties. Moreover, we study the fixed point of modified Jerk map with the form $\mathrm{Mj}_{a,b} = (y - ax + by^2) = (xy)$ and the general properties of them, so, we find the contracting and expanding area of this map. Also, we determine fixed point attracting repelling or saddle.

Key words: Modified Jerk Map, Fixed Point, Attracting-Expanding Area, dimensional, properties, saddle

INTRODUCTION

Non-linear dynamical happens and covered a large area of engineering, physics, biology, and many other scientific disciplines. There is a large amount of interest in the chaos literature in finding out of chaos in natural and physical system. Poincare way the first to notice the possibility of chaos according to which a deter monistic system to show periodic behavior that depends on the initial condition. There by rendering long term could be expected impossible.

Three dimensional system which is simple and chaotic have been tried to find by Gottlieb (1996), Hoover (1995), Linz (1997), Patidar and Sud (2005) and Posch *et al.* (1986). By sprott (1994) found 19 clear chotic models (one of them is conservative and the lasting are dissipative, mentioned as models A –S) with three-dimensional vector fields that involve of five terms including two non-linearities or of six terms with non-linearity one quadratic.

Hoover (1995), stated that the special case of nose that has been created by Sportt (1994) is simply conservative system (Model A) which is a known Hoover thermostat dynamical system. This system shows a time reversible Hamiltonian chaose (Linz, 1997). Apart from this the others models from B-S are unknown. Moreover, Gottliebl indicated that 'Jerk function' is a noticeable third-order form y=J(Y,.Y,..Y) produces from reorder the sprott’s Model A. This function contains the third derivative of x. According to Gottlieb (1996) study, an exciting question has been an annoyance “what is the simplest jerk function that gives chaos?”.

Linz (1997) shows that there models can be reduce to a a jerk form named original R¨Ossler Model, Lorenz Model and Sprott' Model R. Moreover the difficulty of R¨Ossler and Lorenz Models are higher and imappropirate candidate for the Gottlieb's simplest jerk function.

In Patidar and Sud (2005), the global dynamic of some member of a special family of a dynamic system has been explored by Patidar. The following form represents this dynamic system under consideration:

$$Y + AY \ddot{+} \dot{Y} = G(X)$$

Where the system parameter represented by *A* and the nonlinear function that have three arguments which are one nonlinearity, one system parameter and a constant term represented by G(X).

In this research, we simplified the Jerk map by replacing three dimensional systems to two dimensional systems and include the new parameter so we get the difference (new various) properties of dynamical behavior from Jerk

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map also there exist the same properties of it. Also, we show that the modified Jerk map is a diffeomorphism map and it has two fixed points, we determinate the type of fixed points.

MATERIALS AND METHODS

Preliminaries: "For any map G define from R^2 to R^2, we say G is C^∞, if it's continuous for all $k \in Z_+$ and mixed K-th partial derivatives exist, and we say that G is diffeomorphism map if it is onto, one to one, C^∞ and its inverse is C^∞. Let U be a subset of R^2 and v_0 be any element in R^2. Consider $G: V \to R^2$ be a map. Furthermore, assume that the first partials of the coordinate maps f and gof G exist at s_0. The differential of g at s_0 is the linear map $DG(s_0)$ defined on R^2 by:

$$DG(s_0) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

for all s_0 in R^2. The determinate of $DG(s_0)$ is called the Jerk of G at s_0 and it is denoted by $JG(s_0) = \det DG(s_0)$, so G is said to be area-contracting at v_0 if $|\det DG(v_0)| < 1$, G is said to be area-expanding at v_0 if $|\det DG(v_0)| > 1$.

For any nxn matrix B, the scalar λ is called aneigenvalue of $n \times n$ matrix B if there exists a non zero vector $Z \in R^n$ such that $BZ = \lambda Z$, such an Z is called eigenvector of B corresponding to eigen value λ. Any pair for which $f(pq) = p.g(pq) = q$ is called a fixed point of the two dimensional dynamical system it is repelling fixed point if λ_1 and λ_2 are >1 in absolute value , and it is an attracting fixed point if λ_1 and λ_2 is<1 in absolute value $B \in GL(2, Z)$ with det (B) = ± 1 is called hyperbolic matrix if $|\lambda_i|$ wher λ_i are the eigenvalues of B" (Denny, 1992).

RESULTS AND DISCUSSION

General properties of modified jerk map

Proposition (3.1) Let $MJ: R^2 \to R^2$ be modified Jerk map then MJ has two fixed points:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{a+1}{b} \\ \frac{a+1}{b} \end{pmatrix}$$

Proof: By definition of fixed point:

$$MJ_{a,b} = \begin{pmatrix} y \\ -ax + by^2 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then $x = y$ and $-Ay + By^2 - y = 0, by^2 - (a+1)\,y = 0$, $y\,[by - (a+1)] = 0$ then $y = 0$ or $y = a + 1/b$ and $x = a + 1/b$. Therefor:

$$P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and } P_2 = \begin{pmatrix} \frac{a+1}{b} \\ \frac{a+1}{b} \end{pmatrix}$$

Proposition (3.2): Let $MJ: R^2 \to R^2$ be modified Jerk map then the Jacobin of $MJ_{a,b}$ is a:

Proof:

$$DMJ_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a & 2by \end{pmatrix} \text{ so } DMJ_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ -a & 2by \end{pmatrix} = a$$

Proposition (3.3): The eigen value of:

$DMJ_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are $\lambda_{1,2} = by \mp \sqrt{(by)^2 - a}$ at p_2 , $\forall \begin{pmatrix} x \\ y \end{pmatrix} \in R^2$

Proof: If λ is eigen value of $DMJ_{a,b}$ then must be satisfied the following equation:

$$\begin{vmatrix} -\lambda & 1 \\ -a & 2by - \lambda \end{vmatrix} = 0$$

Then:

$$(-\lambda)(2by - \lambda) + a = 0$$

$$\lambda^2 - 2by\lambda + a = 0$$

And the solution of characteristic equations are:

$$\lambda_{1,2} = \frac{2by \mp \sqrt{(2by)^2 - 4a}}{2}$$

$$\lambda_{1,2} = by \mp \sqrt{(by)^2 - a}$$

Remark:

- It is clear that if $|by| > \sqrt{a}$ then the eigen value of $DMJ_{a,b}$ is real.
- The eigen value of $DMJ_{a,b}$ at fixed point P_1 are $\lambda_{1,2}^0 = \mp\sqrt{-a}$ and $\lambda_{1,2} = a + 1 \mp \sqrt{(a+1)^2 - a}$ for the fixed point P_2.

Proposition (3.4): Let $MJ: R^2 \to R^2$ be modified Jerk map if $a \neq 0$ then $MJ_{a,b}$ is diffeomorphism.

Proof*:* To prove $MJ_{a,b}$ is one to one. Let

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in R$$

Such that:

$$MJ_{a,b}\begin{pmatrix}x_1\\y_1\end{pmatrix} = MJ_{a,b}\begin{pmatrix}x_2\\y_2\end{pmatrix}$$

Then:

$$\begin{pmatrix}y_1\\-ax_1+by_1^2\end{pmatrix} = \begin{pmatrix}y_2\\-ax_2+by_2^2\end{pmatrix}$$

Then:

$$y_1 = y_2 \text{ and } -ax_1+by_1^2 = -ax_2+by_2^2$$

Hence, $-ax_1 = -ax_2$ then $x_1 = x_2$,$MJ_{a,b}$ is C^∞:

$$MJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}y\\-ax+by^2\end{pmatrix}$$

then all first partial derivative exist and continuous, note that:

$$\frac{\partial^n f(x.y)}{\partial x^n} = 0 \ \forall\, n \in N$$

And

$$\frac{\partial^n f(x.y)}{\partial y^n} = 0 \ \forall\, n \geq 2$$

$$\frac{\partial^n g(x.y)}{\partial x^n} = 0 \ \forall\, n \geq 3$$

We find all its $MJ_{a,b}$ exist Kth partial derivative exist and continuous for all k. $MJ_{a,b}$ is onto:

$$\text{Let } \begin{pmatrix}v\\w\end{pmatrix} \in R^2$$

Such that:

$$y = v \text{ and } -Ax+By^2 = w$$

So:

$$x = \frac{w-bv^2}{-a}$$

Then, there exist:

$$\begin{pmatrix}\dfrac{bv^2-w}{a}\\v\end{pmatrix} \in R^2$$

Such that:

$$MJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}v\\w\end{pmatrix}$$

Then $MJ_{a,b}$ is onto $MJ_{a,b}$ has an invers.

Remark: if $a = 0$ then:

$$MJ_{a,b} = \begin{pmatrix}y\\by^2\end{pmatrix}$$

So:

$$\text{Ker}\,(MJ_{a,b}) = \begin{pmatrix}x\\y\end{pmatrix} : x.y \in R$$

Then $MJ_{a.b}$ is not one to one hence $MJ_{a,b}$ is not diffeomorphism**.**

Proposition (3.5):

$$DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}$$

is non hyperbolic if $\mp\frac{\sqrt{a}}{b}$ also it is non hyperbolic matrix if $a = -1$

Proof: Let $y = \mp\frac{\sqrt{a}}{b}$ then $b^2y^2 = a$ and $b^2y^2 - 2by\sqrt{a} + a = (by)^2 - \sqrt{a}$.

Hence:

$$x = \left|by \mp \sqrt{(by)^2 - a}\right| = 1$$

So $\lambda_1 = \lambda_2 = 1, \lambda_{1,2}^0 = \mp\sqrt{-a}$ the eigenvalue at p_1.

If $a = -1$ then $\lambda_{1,2}^0 = \mp 1$.

Proposition 3.6: For all:

$$\begin{pmatrix}x\\y\end{pmatrix} \in R^2, y \neq \frac{\sqrt{a}}{b}$$

And:

$$|by| > \sqrt{a}\ , DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}$$

is hyperbolic matrix if and only if $|a| = 1$.

Proof: Since $DMJ_{a,b} \in GL(2.z)$ and:

$$\det\left(DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\right) = |a|$$

then $a = 1$ or $a = -1$

Conversely: Let $a = 1$, since

$$\det\left(DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\right) = a$$

we have $\lambda_1\lambda_2 = a = 1$ Then $\lambda_1 = \frac{1}{\lambda_2}$ if $\lambda_1 > 1$ $\lambda_2 < 1$ or $\lambda_1 < 1$ then $\lambda_2 > 1$. So, $DMJ_{a,b}$ is hyperbolic matrix.

Proposition (3.7): if $|a| < 1$ then $DMJ_{a,b}$ is area–contracting and its area expanding if $|a| > 1$.

Proof: if $|a| < 1$ then by proposition (Jorobian):

$$\left|\det\left(DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\right)\right| < 1$$

that is $DMJ_{a,b}$ is area–contracting map and if $|a| > 1$ then

$$\left|\det\left(DMJ_{a,b}\begin{pmatrix}x\\y\end{pmatrix}\right)\right| > 1$$

This implies that $MJ_{a,b}$ is an area-expending.

Fixed Point Properties of Modified Jerk Map

Proposition (4-1): If $|a| > 1$ thenThe fixed point:

$$\begin{pmatrix}\frac{a+1}{b}\\\frac{a+1}{b}\end{pmatrix}$$

of modified Jerk map is repelling where $a > 1$. The fixed point:

$$\begin{pmatrix}0\\0\end{pmatrix}$$

of modified Jerk map is repelling where $a < -1$.

Proof (1): Since:

$$+1 > a, \sqrt{(a+1)^2 - a} > \sqrt{(a-2)^2}$$

Then:

$$a + 1 \mp \sqrt{(a+1)^2 - a} > a \mp \sqrt{(a-2)^2}$$

By proposition (3-1and 3-3)Thuse $|\lambda_1| = |\lambda_2| > 1$. Then:

$$\begin{pmatrix}\frac{a+1}{b}\\\frac{a+1}{b}\end{pmatrix}$$

is repelling fixed point.

Proof (2): Since $a < -1$ then we have $\sqrt{-a}>1$ by proposition (2-1and 2-3) $|\lambda_1^0| = |\lambda_2^0| > 1$. Hence:

$$\begin{pmatrix}0\\0\end{pmatrix}$$

is repelling fixed point.

Proposition (4-2): if $|a| < 1$then, $MJ_{a,b}$ has attracting fixed point:

$$\begin{pmatrix}0\\0\end{pmatrix}$$

and saddle fixed point:

$$\begin{pmatrix}\frac{a+1}{b}\\\frac{a+1}{b}\end{pmatrix} \text{where } a > 1$$

$MJ_{a,b}$ has attracting fixed point:

$$\begin{pmatrix}\frac{a+1}{b}\\\frac{a+1}{b}\end{pmatrix} \text{where } a < 1$$

Proof (1)*:* Since, $-1 < a$ then we have $\sqrt{-a} < 1$ and $|\lambda_1^0| = |\lambda_2^0| < 1$

By proposition (3-1) and (3-3). Hence:

$$\begin{pmatrix}0\\0\end{pmatrix}$$

is attracting fixed point.

Now to show that P_2 is a saddle fixed point by proposition (2-3), we have two eigenvalue:

$$\lambda_1 = a + 1 + \sqrt{(a+1)^2 - a},\ \lambda_2 = a + 1 - \sqrt{(a+1)^2 - a} \quad (1)$$

Since:

$$\sqrt{a^2 + a + 1} < \sqrt{(a+2)^2} = |a+2| = -(a+2) \quad (2)$$

Hence:

$$a + 1 + \sqrt{a^2 + a + 1} < 1 \quad (3)$$

So:

$$\left|a + 1 + \sqrt{a^2 + a + 1}\right| = |\lambda_1| < 1 \quad (4)$$

For λ_2 since $-1 < a < 0$ we have:

$$\sqrt{a^2 + a + 1} > \sqrt{a^2} = |a| = -a \quad (5)$$

By adding (a+1) for both side we get:

$$a+1-\sqrt{(a+1)^2 - a} > 1 \text{ so } |\lambda_2| > 1 \quad (6)$$

From Eq. 3-5 and by proposition (3-1) we obtain

$$\begin{pmatrix}\frac{a+1}{b}\\\frac{a+1}{b}\end{pmatrix}$$

is a saddle fixed point.

Proof (2): Since:

$$\sqrt{a^2 + a + 1} < \sqrt{(a+1)^2} = |a+1| = -(a+1).$$

hence by adding (a+1) for both side, we get:

$$a+1+\sqrt{a^2+a+1}<1 \qquad (6)$$

$\sqrt{a^2+a+1}>\sqrt{a^2}=|a|=a$ So, $-\sqrt{a^2+a+1}<-a$

By adding (a+1) for both side we get:

$$a+1-\sqrt{(a+1)^2-a}<1 \qquad (7)$$

From Eq. 6 and 7 and by proposition (3-1) we obtain

$$\begin{pmatrix} \frac{a+1}{b} \\ \frac{a+1}{b} \end{pmatrix}$$

is attracting fixed point.

CONCLUSIONS

In this study, a simplified version of Jerk map is presented. Different properties of dynamical behavior is acquired by replacing three dimensional systems to two dimensions by adding new parameters that have the same properties.

For this class of dynamical system the improved Jerk Map becomes a representative. The main mathematical properties of the modified Jerk Map is formulated. Two fixed point is gotten P_1 and P_2 when $b \neq 0$. The eignvalues of the modified Jerk map is $\lambda_{1,2}^0 = \mp\sqrt{-a}$ at p_1 and $\lambda_{1,2} = a+1\mp\sqrt{(a+1)^2-a}$ at P_2 if $|a|<1$ then $DMJ_{a,b}$ is an area-contraction map and if $|a|>1$ then $DMJ_{a,b}$ is an area expending map. Moreover, the improve Jerk map is shown to be a diffeomorphism map.

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