Study of *Hpre*-open sets in topological spaces

Cite as: AIP Conference Proceedings **2292**, 040008 (2020); https://doi.org/10.1063/5.0030807
Published Online: 27 October 2020

Mustafa Hasan Hadi, and May Alaa Abdul-khaleq AL-Yaseen

Study Of $Hpre$-Open Sets In Topological Spaces

Mustafa Hasan Hadi[1,a)] and MayAlaa Abdul-khaleq AL-Yaseen[2, b)]

[1,2]University of Babylon, College of Education for Pure Scineces, Department of Mathematics, Iraq.

Corresponding Author: [a)]pure.mustafa.hassan@uobabylon.edu.iq
[b)]may_alaa2004@yahoo.com

Abstract: The concept main of paper is to connected between two definitions pre-open sets and g-open sets in the topological spaces is called $Hpre$-open sets, study restudy relations them, also present a new definitions, namely $Hpre$-continous, $Hpre$-open functions, $Hpre$-closed functions and have proved some theorems about these concept. Finally, we have present the definitions and results of separation axioms by $Hpre$-open sets,namely $Hpre - T_1$spaces , $Hpre - T_2$spaces and studied some of its applications.

INTRODUCTION

The great role played and because is very important entered into all the topics of general topology of which continuity, separation axioms, connected,etc. The first to introduce the definition of generalized closed sets (g-closed) is Levine [12] in 1970. Dunham and N. Levine [9] in1980 gave a new results about the generalized closed sets, and studied some of the immovable of this new topology, to learn the characteristics of more recipes (see,[1], [7], [8]). In 1982 Mashouretal. [14-15] introduced the concept of pre-open sets and pre-continuous in topological spaces. In 1986 Andrijevic[2] used ideas pre-open, to define a new types of sets named$semi - pre$ open sets and used it to offer a new kind of continuity. [11] Kar and Bhattacharyya introduce of weak separation axioms by notion pre-open sets, to learn more about pre-open sets (see [4], [6], [10]).Throughout our paper $(X,\tau), (Y,\sigma)$ and (Z,ρ)(simply X, Y and Z)stands for a topological space and no separation axioms. A subset $\mathcal{H}$ of X, the interior and the closure of $\mathcal{H}$ are symbol by$Int(\mathcal{H})$ and $Cl(\mathcal{H})$, respectively. In section 3 introduced a definition $Hpre$- closed sets and show some results. In section 4 stududied$Hpre$- open sets, showed some relationships [16-19]. In section 5 took some applications about $Hpre$- open in the continuity and separation axioms [20-22].

PRELIMINARIES

Definition 2.1. Let (X,τ) be a topological space and $A \subseteq X$. Then
(1) Ais pre-open [14] set if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. And pre-closed if $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq A$.
(2) The g-closure of A in X [7] is the intersection for any$g$-closed containing A and is denoted by $Cl_g(A)$ and the union of all g-open sets contain in A [7] is called g-interior and is denoted by $Int_g(A)$. Now we need the following notation

- $preC(X)$ (resp. $preO(X)$) denotes the family of all pre-closed sets (resp. pre-open sets).
- $GC(X)$ (resp. $GO(X)$) denotes the family of all g-closed sets (resp. g-open sets).

Definition 2.2. A function $f: X \to Y$ is called:
(1) pre-continuous [14] (resp. g-continuous [3]) if $f^{-1}(U)$ is pre-open (resp.g-open) in X for each open subset U in Y.

Proceedings of the 2020 2nd International Conference on Sustainable Manufacturing, Materials and Technologies
AIP Conf. Proc. 2292, 040008-1–040008-8; https://doi.org/10.1063/5.0030807
Published by AIP Publishing. 978-0-7354-4024-1/$30.00

(2) pre-open [14] (resp. pre-closed [14]) if $f(U)$ is pre-open (resp. pre-closed) in Y for open open (resp. closed) subset U in X.
(3) g-open [13] (resp. g-closed [13]) if $f(U)$is g-open (resp. g-closed) in Y for each open (resp. closed) subset U in X.

(1) pre-T_1 [15,11] (resp.g-T_2 [5]) if for each $x \neq y \in X$, there exist a pre-open (resp. g-open) set U in X, such that $x \in U, y \notin U$ and there exsit a pre-open (resp. g-open) set V in X, such that $y \in V, x \notin V$.
(2) pre-T_2 [15,11] (resp. g-T_2 [5]) if for each $x \neq y \in X$, $\exists$ a pre-open (resp. g-open) U and V sets such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

$Hpre$-CLOSED SETS

Now we define the set$Hpre$-closed also introduce the properties, examples and theorems.

Definition 3.1. A subset $\mathcal{H}$ of a space X is called $Hpre$-closed set if $Cl_g\big(Int(\mathcal{H})\big) \subseteq \mathcal{H}$.$HpreC(X)$is means of all $Hpre$-closed sets.

Proposition 3.2. If B is an g-closed subset of $X$and$Cl_g\big(Int(B)\big) \subseteq A \subseteq B$,therefore$A$ is $Hpre$-closed set.
Proof. Since B is$g$-closed, $Cl_g(B) = B$. Then $Cl_g\big(Int(A)\big) \subseteq Cl_g\big(Int(B)\big) \subseteq A$, so A is $Hpre$-closed set.
Remark 3.3. The converse of Lemma (3.2) is not true as shown by the following example.
Example 3.4.Let $X = \{x_1, x_2, x_3\}$ and $\tau = \{X, \phi, \{x_1\}, \{x_1, x_2\}\}$. Then $\tau^c = \{X, \phi, \{x_3\}, \{x_2, x_3\}\}$, $preC(X) = \{X, \emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$, $preO(X) = \big\{X, \emptyset, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\big\}$, $GC(X) = \{X, \phi, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}\}$, $GO(X) = \{X, \emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}\}$, $HpreC(X) = \{X, \emptyset, \{x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}\}$, $HpreO(X) = \{X, \emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_1, x_3\}\}$. So $\{x_3\} \in HpreC(X)$ and $\{x_1, x_3\} \in GC(X)$, but $Cl_g(Int\{x_1, x_3\}) = \{x_1, x_3\} \not\subset \{x_3\} \subset \{x_1, x_3\}$.
Theorem 3.5.In any topological space$(X, \tau)$the statements are verified
(1) Any$g$-closed subset of X is $Hpre$-closed set.
Proof. (1) Suppose that A is pre-closed subset of X, since any closed is g-closed, $Cl_g(A) \subseteq Cl(A)$[7, Lemma 3.3], then $Cl(Int(A)) \subseteq A$, this implies $Cl_g(Int(A)) \subseteq Cl(Int(A)) \subseteq A$. Then A is $Hpre$-closed set.
(2) Since $Int(A) \subseteq A$ and suppose that A is g-closed then $Cl_g(A) = A$ [7, Theorem 3.7], so $Cl_g(Int(A)) \subseteq Cl_g(A) = A$. Then A is $Hpre$-closed set.
Remark 3.6.The opposite of Theorem 3.5 is nottrue. In Example (3.4) note that, $\{x_1, x_3\} \in HpreC(X)$, but $\{x_1, x_3\} \notin preC(X)$ and $\{x_2\} \in HpreC(X)$, but $\{x_2\} \notin GC(X)$.
The above outcomes can be outlined in the accompanying chart, however the opposite is nottrue.
Theorem 3.7. If $\{G_i : i \in \Delta\}$ is a collection of $Hpre$-closed sets of a space (X, τ), then $\bigcap_{i\in\Delta} G_i$ is $Hpre$-closed set.

-closed set pre

Proof. Since $Cl_g\big(Int(G_i)\big)$subset $G_i$for every $i \in \Delta$, and $\bigcap_{i\in\Delta} G_i$ subset G_i , for every i, $Int(\bigcap_{i\in\Delta} G_i) \subseteq Int(G_i)$. So $Int(\bigcap_{i\in\Delta} G_i) \subseteq \bigcap_{i\in\Delta} Int(G_i)$. Hence $Cl_g(Int(\bigcap_{i\in\Delta} G_i)) \subseteq Cl_g(\bigcap_{i\in\Delta} Int(G_i)) \subseteq \bigcap_{i\in\Delta} Cl_g(Int(G_i)) \subseteq \bigcap_{i\in\Delta} G_i$, therefore $\bigcap_{i\in\Delta} G_i$is $Hpre$-closed set.
Remark 3.8. The union of two $Hpre$-closed sets it is not necessary be $Hpre$-closed set. As can be seen in the following example.
Example3.9.Let $X = \{x_1, x_2, x_3\}$ and $\tau = \{X, \phi, \{x_2, x_3\}\}$. Then $\tau^c = \{X, \phi, \{x_1\}\}$, $preC(X) = HpreC(X) = \{X, \emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}\}$, $preO(X) = HpreO(X) = \{X, \emptyset, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}\}$,$GC(X) = \{X, \phi, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\}$, $GO(X) = \{X, \emptyset, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$. Where$\{x_2\}$ and $\{x_3\}$ are $Hpre$-closed sets, but $\{x_2\} \cup \{x_3\} = \{x_2, x_3\}$ is not $Hpre$-closed set.
Corollary 3.10. The intersection of two $Hpre$-closed set is $Hpre$-closed set.
Proof. Direct from Theorem 3.5(1) and Theorem 3.7.

Definition 3.12. Let $\mathcal{H} \subseteq X$. The $Hpre$-closure for$\mathcal{H}$ is symbolic $Cl_{pre}^{H}(\mathcal{H})$ is $Cl_{pre}^{H}(\mathcal{H}) = \cap \{G: \mathcal{H} \subseteq G, G \in HpreC(X)\}$.
Proposition 3.13. A subset $\mathcal{H}$ of a spaces$X$. Then $\mathcal{H}$ is $Hpre$-closed set in Xiff$Cl_{pre}^{H}(\mathcal{H}) = \mathcal{H}$.
Proof. Let $\mathcal{H}$ is $Hpre$-closed set. So$Cl_{pre}^{H}(\mathcal{H}) = \mathcal{H}$. Conversely, let$Cl_{pre}^{H}(\mathcal{H}) = \mathcal{H}$. From Theorem 3.7 then $\mathcal{H}$ is $Hpre$-closed set.

$Hpre$-OPEN SETS

Now we will study $Hpre$-open sets, review the basic properties and some important theorems.
Lemma4.2. A subset A of a space X, then
(1) $X \setminus Cl_g(X \setminus A) = Int_g(A)$.
(2) $X \setminus Int_g(X \setminus A) = Cl_g(A)$.
Proof. Easy.
Theorem4.3. Let $A \subseteq X$ is $Hpre$-open iff$A \subseteq Int_g(Cl(A))$.
Proof. Suppose thatA be Hpre-open subset of X. So$X \setminus A$ is Hpre-closed also$\mathrm{Cl_g}(Int(\mathrm{X} \setminus \mathrm{A})) \subseteq X \setminus A$.Then$\mathrm{A} \subseteq \mathrm{Int_g}(\mathrm{Cl(A)})$.Conversely, let$A \subseteq Int_g(Cl(A))$. So $\mathrm{X} \setminus \mathrm{Int_g}\big(\mathrm{Cl(A)}\big) \subseteq \mathrm{X} \setminus \mathrm{A}$, therefore $\mathrm{Int_g}(\mathrm{Cl}(\mathrm{X} \setminus \mathrm{A})) \subseteq \mathrm{X} \setminus \mathrm{A}$, then $X \setminus A$ is Hpre-closed, so A is Hpre-open.
Directly from Lemma 4.2 and Lemma 3.2.
Remark4.5. Something contrary to above lemma isn't valid. See Example (3.4) note that, $\{\mathrm{x_1}, \mathrm{x_2}\} \in \mathrm{H}preO(X)$and $\{x_2\} \in GO(X)$, but $\{x_2\} \subset \{x_1, x_2\} \not\subset \{x_2\}$.
Theorem4.6.In any topological space $(X, \tau)$the statements are verified
(1) Anypre-open is $Hpre$-open set.
(2) Any$g$-open is $Hpre$-open set.
Proof. DirectlybyTheorem 3.5.
Remark 4.7. Something contrary to above theorem isn't valida show from Example 3.4, since $\{x_2\} \in HpreO(X)$, but $\{x_2\} \notin preO(X)$ and $\{x_1, x_3\} \in HpreO(X)$, but $\{x_1, x_3\} \notin GO(X)$.
The above outcomes can be outlined in the accompanying chart, however the opposite is not true

-open set pre

Theorem 4.8. If $\{F_i: i \in \Delta\}$ is a collection of $Hpre$-opn sets of a space (X, τ), then $\cup_{i \in \Delta} F_i$ is $Hpre$-oen set.
Proof. Obvious. From Theorem 3.7.
Remark 4.9. The intersection of two $Hpre$-open sets is not necessary by $Hpre$-oen set. In Example 3.9, since two $Hpre$-open sets are $\{x_1, x_2\}$ and $\{x_1, x_3\}$ but $\{x_1, x_2\} \cap \{x_1, x_3\} = \{x_1\}$ is not $Hpre$-open sets.
Corollary4.10.In anyA topological space(X, τ). Then

(1) The intersection of $Hpre$-open set and a pre-open set is $Hpre$-open set.
(2) The intersection of an $Hpre$-open set and a g-open set is $Hpre$-open set.
Proof. (1) Directly from Theorem 4.6 (1) and Theorem 4.8.
(2) Directly from Theorem 4.6 (2) and Theorem 4.8.
Definition 4.11. A subset A of X. The $Hpre$-interior of A is symbol by $Int_{pre}^{H}(A)$ is $Int_{pre}^{H}(A) = \cup \{F: A \subseteq F, F \in HpreO(X)\}$.
Lemma4.12. A subset A of a space X, then
(1) $X \setminus Cl_{pre}^{H}(A) = Int_{pre}^{H}(X \setminus A)$.
(2) $X \setminus Int_{pre}^{H}(A) = Cl_{pre}^{H}(X \setminus A)$.
Proof. Easy.
Theorem4.13. $A \subseteq X$. SoA is $Hpre$-open set iff$Int_{pre}^{H}(A) = A$.
Proof. Easy.
Proposition 4.14. Any a subset A of (X, τ), then

(1) $A \cap Int_g(Cl(A))$ is $Hpre$-open set.
(2) $A \cup Cl_g(Int(A))$ is $Hpre$-closed set.
Proof. (1) $Int_g(Cl(A \cap Int_g(Cl(A)))) = Int_g\left(Cl(A) \cap Int_g(Cl(A))\right) = Int_g(Cl(A))$. Then $A \cap Int_g(Cl(A)) = A \cap Int_g(Cl(A \cap Int_g(Cl(A)))) \subseteq Int_g(Cl\left(A \cap Int_g(Cl(A))\right))$. So $A \cap Int_g(Cl(A))$ is $Hpre$-open set.
(2) From (1) we have $X \setminus (A \cup Cl_g(Int(A)) = (X \setminus A) \cap Int_g(Cl(X \setminus A))$ is $Hpre$-open set. This also means $A \cup Cl_g(Int(A))$ is $Hpre$-closed set.
Proposition 4.15.A subset A of a space (X, τ), then
(1) $Int^H_{pre}(A) = A \cap Int_g(Cl(A))$.
(2) $Cl^H_{pre}(A) = A \cup Cl_g(Int(A))$.
Proof. (1) Suppose that $M = Int^H_{pre}(A)$. It is clear that M is $Hpre$-open set and $M \subseteq A$. Since M is $Hpre$-open set, $M \subseteq Int_g(Cl(M)) \subseteq Int_g(Cl(A)$. This proves it $M \subseteq A \subseteq Int_g(Cl(M))$. By Proposition 4.14, $A \cap Int_g(Cl(A))$ is $Hpre$-open set. By the definition of $Int^H_{pre}(A)$, $A \cap Int_g(Cl(A)) \subseteq M$. Therefore $M = A \cap Int_g(Cl(A))$. So $Int^H_{pre}(A) = A \cap Int_g(Cl(A))$.
(2) By Lemma 4.12 we will get,$Cl^H_{pre}(A) = X \setminus Int^H_{pre}(X \setminus A) = X \setminus ((X \setminus A) \cap Int_g(Cl(A)))$, by (1) $= X \setminus (X \setminus A) \cup \left(X \setminus Int_g(Cl(X \setminus A))\right) = A \cup Cl_g(Int(A))$.

APPLICATIONS$Hpre$-OPEN SETS

In this section we present $Hpre$-continuous, $Hpre$-open and $Hpre$-closed functions with by relations, propositions and theorems about them.
Definition 5.1.A function $f: X \to Y$ is said to be $Hpre$-continuous if $f^{-1}(G)$ is $Hpre$-open for every open set G in Y.
Theorem5.2. For A function $f: X \to Y$, then the following properties are verified:
(1) Anypre-continuous function is $Hpre$-continuous function.
(1) Any$g$-continuous function is $Hpre$-continuous function.
Proof. Directly from Theorem 4.6.
Now $\tau^c = \{X, \phi, \{x_2, x_3\}\}$,$preC(X) = \{X, \phi, \{x_2\}, \{x_3\}, \{x_2, x_3\}\}$, $preO(X) = \{X, \phi, \{x_1\}, \{x_1, x_2\}, \{x_1, x_3\}\}$, $GC(X) = \{X, \phi, \{x_2\}, \{x_1, x_3\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}\}$, $GO(X) = \{X, \emptyset, \{x_1\}, \{x_3\}, \{x_2\}, \{x_1, x_3\}, \{x_1, x_2\}\}$, $HpreC(X) = HpreO(X) = P(X)$.Since$\{y_3\}$ is open in Y, $f^{-1}(\{y_3\}) = \{x_3\} \in HpreO(X)$, but $\{x_3\} \notin preO(X)$. Then f is $Hpre$-continuous function, but not pre-continuous function.
(2) Assume that (X, τ) and $(Y, \sigma)$are topological spaces in (1) and $f: X \to Y$ defined by $f(x_1) = y_1$, $f(x_2) = f(x_3) = y_3$. Note that $\{y_3\}$ is open in Y, $f^{-1}(\{y_3\}) = \{x_2, x_3\} \in HpreO(X)$, but $\{x_2, x_3\} \notin GO(X)$. Then f is $Hpre$-continuous function, but not$g$-continuous function.
The above outcomes can be abridged in the accompanying chart, yet the oppositeis not true.

-continuous function pre

Theorem5.5.A function $f: X \to Y$is $Hpre$-continuous function iff satisfying one of the following properties:
(1) The inverse image of every closed subset$F$ in Y is $Hpre$-closed in$X$.
(2)$Cl^H_{pre}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$, $B \subseteq Y$.
(3) $f^{-1}(Int(B)) \subseteq Int^H_{pre}(f^{-1}(B))$, $B \subseteq Y$.
Proof. (1) $\Rightarrow$) Suppose that f is $Hpre$-continuous. To proof $X \setminus f^{-1}(F) \in HpreO(X)$, for every $Y \setminus F \in Y$. Let F is closed set in Y. This implies $Y \setminus F$ is open in Y and $f^{-1}(Y \setminus F) \in HpreO(X)$. But $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$. Then $X \setminus f^{-1}(F) \in HpreO(X)$. So $f^{-1}(F)$ is $Hpre$-closed in X. $\Leftarrow$) Suppose that $X \setminus f^{-1}(F) \in HpreO(X)$, for every $Y \setminus F \in Y$. To proof f is $Hpre$-continuous, let V is open in Y so $Y \setminus V$ is closed in Y this imples $f^{-1}(Y \setminus F) \in HpreC(X)$ then $X \setminus f^{-1}(Y \setminus V) \in HpreO(X)$. But, $X \setminus f^{-1}(Y \setminus V) = X \setminus \left(X \setminus f^{-1}(V)\right) = f^{-1}(V) \in HpreO(X)$. Therefore f is $Hpre$-continuous.

(2) ⇒)Suppose that f is $Hpre$-continuous. Let $B \subseteq Y$, $B \subseteq Cl(B)$ then $f^{-1}(B) \subseteq f^{-1}(Cl_{pre}^{H}(B))$. Since $Cl(B)$ is a closed set in Y by (1) $f^{-1}(Cl_{pre}^{H}(B)) \in HpreC(X)$ and therefore $Cl_{pre}^{H}(f^{-1}(B)) = f^{-1}(Cl(B))$. Now $B \subseteq Cl(B)$implies that $f^{-1}(B) \subseteq f^{-1}(Cl(B))$ so $Cl_{pre}^{H}(f^{-1}(B)) \subseteq Cl_{pre}^{H}(f^{-1}(Cl(B))) = f^{-1}(Cl(B))$. ⇐) Let the condition hold and let F be any closed set in Y so that $Cl(F) = F$. By hypothesis $Cl_{pre}^{H}(f^{-1}(F)) \subseteq f^{-1}\big(Cl(F)\big) = f^{-1}(F)$. But, $f^{-1}(F) \subseteq Cl_{pre}^{H}(f^{-1}(F))$ and $Cl_{pre}^{H}(f^{-1}(F)) = f^{-1}(F)$. Hence $f^{-1}(F) \in HpreC(X)$. It follows from (1)that$f$ is $Hpre$-continuous.

(3)⇒)Let f is $Hpre$-continuous. Hence$Int(B)$ is open of$Y$, $f^{-1}(Int(B)) \in HpreO(X)$ and therefore $Int_{pre}^{H}(f^{-1}(Int(B)) = f^{-1}(Int(B))$. Now $Int(B) \subseteq B$ this implies $f^{-1}(Int(B)) \subseteq f^{-1}(B)$ therefore $Int_{pre}^{H}(f^{-1}(Int(B))) \subseteq Int_{pre}^{H}(f^{-1}(B))$, so $f^{-1}(Int(B)) \subseteq Int_{pre}^{H}(f^{-1}(B))$. ⇐) Let the condition hold and let G be any open set in Y, so that $Int(G) = G$. By hypothesis $f^{-1}(Int(G)) \subseteq Int_{pre}^{H}(f^{-1}(G))$. But, $Int_{pre}^{H}(f^{-1}(G)) \subseteq f^{-1}(G)$, so $Int_{pre}^{H}(f^{-1}(G)) = f^{-1}(G)$. Therefore $f^{-1}(G) \in HpreO(X)$. Then f is $Hpre$-continuous.

Definition 5.6.A function $f: X \to Y$ is called$Hpre$-open (resp. $Hpre$-closed) if $f(G)$ is $Hpre$-open (resp. $Hpre$-closed) in Y for every open (resp. closed) set G in X.

Theorem5.7. For A function $f: X \to Y$, then the following properties are verified:

(1) Every pre-open function is $Hpre$-open function.

(1) Any$g$-open function is $Hpre$-open function.

Proof. Directly from Theorem 4.6.

Remark 5.8.(1)$Hpre$-open function does not have to be pre-open function(see Example 5.9(1)).

(2)$Hpre$-open function does not have to be g-open function (see Example 5.9(2)).

Example5.9.(1)Let $X = \{x_1,\ x_2, x_3\}$ with $\tau = \{X, \phi, \{x_2\}\}$ and $Y = \{y_1,\ y_2, y_3\}$ with $\sigma = \{Y, \phi, \{y_2, y_3\}, \{y_2\}\}$. A function $f: X \to Y$ be a function defined by $f(x_1) = y_2$, $f(x_2) = y_3$ and $f(x_3) = y_1$. Now $\sigma^c = \{Y, \phi, \{y_1\}, \{y_1, y_3\}\}$,$preO(Y) = \{Y, \phi, \{y_2\}, \{y_1, y_2\}, \{y_2, y_3\}\}$, $preC(Y) = \{Y, \phi, \{y_1\}, \{y_3\}, \{y_1, y_3\}\}$, $GO(Y) = \{Y, \phi, \{y_2\}, \{y_3\}, \{y_2, y_3\}\}$, $GC(Y) = \{Y, \phi, \{y_1\}, \{y_1, y_2\}, \{y_1, y_3\}\}$, $HpreO(Y) = \{Y, \phi, \{y_2\}, \{y_3\}, \{y_1, y_2\}, \{y_2, y_3\}\}$ and $HpreC(Y) = \{Y, \phi, \{y_1\}, \{y_3\}, \{y_1, y_2\}, \{y_1, y_3\}\}$ Since $\{x_2\}$ is open in X, $f(\{x_2\}) = \{y_3\} \in HpreO(Y)$, but $\{y_3\} \notin preO(Y)$. Then f is $Hpre$-open function, but not pre-open function.

(2) Let $X = \{x_1,\ x_2, x_3\}$ with $\tau = \{X, \phi, \{x_1, x_3\}\}$ and $Y = \{y_1,\ y_2, y_3\}$ with $\sigma = \{Y, \phi, \{y_1, y_2\}\}$. A function $f: X \to Y$ be a function defined by $f(x_1) = y_2$, $f(x_2) = y_1$ and $f(x_3) = y_3$. Now $\sigma^c = \{Y, \phi, \{y_3\}\}$, $GO(Y) = \{Y, \phi, \{y_1\}, \{y_2\}, \{y_1, y_2\}\}$, $GC(Y) = \{Y, \phi, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}\}$, $preO(Y) = HpreO(Y) = \{Y, \phi, \{y_1\}, \{y_2\}, \{y_1, y_3\}, \{y_1, y_2\}, \{y_2, y_3\}\}$ and $preC(Y) = HpreC(Y) = \{Y, \phi, \{y_1\}, \{y_3\}, \{y_2\}, \{y_1, y_3\}, \{y_2, y_3\}\}$ Since $\{x_1, x_3\}$ is open in X, $f(\{x_1, x_3\}) = \{y_2, y_3\} \in HpreO(Y)$, but $\{y_2, y_3\} \notin preO(Y)$. Then f is $Hpre$-open function, but not g-open function.

The above outcomes can be condensed in the accompanying chart, yet the oppositeis not true.

-open function pre

Theorem5.10. For a function $f: X \to Y$, then the following properties are equivalent:

(1) f is $Hpre$-open function.

(2) For every $x \in X$ and every neighborhood V of x, $\exists Hpre$-open set $U \subseteq Y$ containing $f(x)$where$U \subseteq f(V)$.

Proof. (1)⇒(2) Suppose that$x \in X$ and V is nbd of x, therefore there exists open $W \subseteq X$s.t.$x \in W \subseteq V$. Now$U = f(W)$. Since f is $Hpre$-open function, $f(W) = U \in HpreO(Y)$ and so $f(x) \in U \subseteq f(V)$.

(2)⇒(1) Clearly.

Theorem5.11. Let $f: X \to Y$ be $Hpre$-closed (resp. $Hpre$-open) function and $U \subseteq Y$. If $A \subseteq X$ is an open (resp. closed) set containing $f^{-1}(U)$, then there exists $Hpre$-open (resp. $Hpre$-closed) set $V \subseteq Y$ containing U such that $f^{-1}(V) \subseteq A$.

Proof. Suppose that $U \subseteq Y$ and let A be an open (resp. closed) subset of X such that $f^{-1}(U) \subseteq A$. Then $V = Y \setminus f(X \setminus V)$ is $Hpre$-open (resp. $Hpre$-closed) set containing U and $f^{-1}(V) \subseteq A$.

Corollary5.12.For every $A \subseteq Y$ and $f: X \to Y$ is $Hpre$-open function, then $\mathrm{f}^{-1}(\mathrm{Cl}_{\mathrm{pre}}^{\mathrm{H}}(\mathrm{A})) \subseteq \mathrm{Cl}(\mathrm{f}^{-1}(\mathrm{A}))$.

Proof. Since $Cl(f^{-1}(A))$ is closed containing $f^{-1}(A)$ for any $A \subseteq Y$. By Theorem 5.11, then exists $Hpre$-closed set $V \subseteq Y$, $A \subseteq V$ such that $f^{-1}(V) \subseteq Cl(f^{-1}(A))$. Therefore $f^{-1}(Cl^{H}_{pre}(A)) \subseteq f^{-1}(Cl^{H}_{pre}(V)) \subseteq f^{-1}(V) \subseteq Cl(f^{-1}(A))$.

Theorem5.13. A function$f: X \to Y$ is $Hpre$-openiff$f(Int(B)) \subseteq Int^{H}_{pre}(f(B))$ for any B of X.

Proof. Let$f: X \to Y$ is $Hpre$-open ,$B \subseteq X$. So$Int(B)$ is open of$X$ and $f(Int(B))$ is $Hpre$-open set contained by$f(B)$. Hence$f(Int(B)) \subseteq Int^{H}_{pre}(f(B))$. Now , let $f(Int(B)) \subseteq Int^{H}_{pre}(f(B))$of any B of X and V is open of$X$. Hence$Int(V) = V$, $f(V) \subseteq Int^{H}_{pre}(f(V))$. Therefore $f(V) = Int^{H}_{pre}(f(V))$. By Theorem 4.13 $f(V)$ is $Hpre$-open function.

Theorem5.15. If $f: X \to Y$ is bijective functionthen the statements are equivalent:

(1) f^{-1} is $Hpre$-continuous.

(2) f is $Hpre$-open.

(3) $f Hpre$-closed function.

Proof. (1)⇒(2) Let G is open in X. Then $X \setminus G$ is closed. Nowf^{-1} is $Hpre$-cntinuous function, $(f^{-1})^{-1}(X \setminus G)$ is $Hpre$-closed in Y. This implies $f(X \setminus G) = Y \setminus f(G)$ is $Hpre$-closed in Y. Then $f(G)$ is $Hpre$-open in Y. Therefore f is $Hpre$-open function.

(2)⇒(3) Assume that that B is closed in X. So $(X \setminus B)$ is open. Since f is $Hpre$-open, $f(X \setminus B)$ is $Hpre$-open. Therefore $f(X \setminus B) = Y \setminus f(B)$ is $Hpre$-open in Y. So $f(B)$ is $Hpre$-closed in Y. Then f is $Hpre$-closed function.

(3)⇒(1) Assume that B is closed set in X. Since f is $Hpre$-open function, $f(B)$ is $Hpre$-closed. This mean $(f^{-1})^{-1}(B)$ is $Hpre$-closed in Y. Then $f^{-1}$is $Hpre$-continuous function.

Remark 5.16. If $f: X \to Y$ and If $g: Y \to Z$ are both $Hpre$-open functions, then the composition $gof(x)$ need not is$Hpre$-open function seen the following example.

Example5.17. Let $X = \{x_1, x_2, x_3\}$ with the topology $\tau = \{X, \phi, \{x_3\}\}$, $Y = \{y_1, y_2, y_3\}$ with the topology $\sigma = \{Y, \phi, \{y_1\}, \{y_1, y_2\}\}$ and $Z = \{z_1, z_2, z_3\}$with the topology$\rho = \{Z, \emptyset, \{z_1, z_2\}\}$ A function $f: X \to Y$ be a function defined by $f(x_1) = y_1$, $f(x_2) = y_3$ and $f(x_3) = y_2$ and $g: Y \to Z$ is defined by $g(y_1) = z_1$, $g(y_2) = z_3$ and $f(y_3) = z_2$. Since $\sigma^c = \{Y, \phi, \{y_3\}, \{y_2, y_3\}\}$,$GO(Y) = \{Y, \phi, \{y_1\}, \{y_2\}, \{y_1, y_2\}\}$, $GC(Y) = \{Y, \phi, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}\}$, $HpreO(Y) = \{Y, \phi, \{y_1\}, \{y_1, y_2\}, \{y_2\}, \{y_1, y_3\}\}$, $HpreC(Y) = \{Y, \phi, \{y_2\}, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}\}$ and $\rho^c = \{Z, \phi, \{z_3\}\}$,$GO(Z) = \{Z, \phi, \{z_1\}, \{z_2\}, \{z_1, z_2\}\}$, $GC(Z) = \{Z, \phi, \{z_3\}, \{z_1, z_3\}, \{z_2, z_3\}\}$, $HpreO(Z) = \{Z, \phi, \{z_1\}, \{z_2\}, \{z_1, z_2\}, \{z_1, z_3\}, \{z_2, z_3\}\}$, $HpreC(Z) = \{Z, \phi, \{z_1\}, \{z_3\}, \{z_2\}, \{z_2, z_3\}, \{z_1, z_3\}\}$. It is clear that, f and g are $Hpre$-open function. Since $\{x_3\}$ is open in X. But $gof(\{x_3\}) = g(\{y_2\}) = \{z_3\}$ is not $Hpre$-open in Z. Then gof is not $Hpre$-open.

Definition 5.18. Let (X, τ) be a topological space. Then the space (X, τ) is called $Hpre - T_1$ space iff for every distinct pair of points x_1 andx_2 in X, there exists $Hpre$-open sets U in X containing x_1 but not x_2, and a $Hpre$-open set V in X containing x_2 but not x_1.

Theorem5.19.In any topological space $(X, \tau)$the statements are verified

(1) Anypre-T_1 is $Hpre - T_1$ space.

(2) Any$g$-T_1 is $Hpre - T_1$ space.

Proof. Easy.

Remark 5.20.(1)$Hpre - T_1$ space does not have to be pre-T_1 space (see Example 5.4(1)).

(2)$Hpre - T_1$space does not have to be g-T_1 space (see Example 3.9).

Theorem5.21. Let (X, τ) be a topological space. Then

(1) X is $Hpre - T_1$ space if and only if for each $x_1 \neq x_2 \in X$, $\{x_1\}$ is pre-closed set and $\{x_2\}$ is $Hpre$-closed set in X.

(2) X is $Hpre - T_1$ space if and only if for each $x_1 \neq x_2 \in X$, $\{x_1\}$ is g-closed set and $\{x_2\}$ is $Hpre$-closed set in X.

Then $x_1 \in U_{x_1} \subset \{x_2\}^c$. Therefore $\{x_2\}^c = \bigcup_{x_{1 \in \{x_2\}^c}} U_{x_1}$, which is $Hpre$-open set and $\{x_2\}$ is $Hpre$-closed set in X. Also$\{x_1\}$$Hpre$-closed set in X. In fact $x_2 \in U_{x_2} \subset \{x_1\}^c$, then $\{x_1\}^c = \bigcup_{x_{2 \in \{x_1\}^c}} U_{x_2}$, which is $Hpre$-open set and $\{x_1\}$ is $Hpre$-closed set in X. Now for the converse, let $x_1 \neq x_2 \in X$, $U_{x_1} = X \setminus \{x_2\}$ is$Hpre$-open set in X containing x_1 but not x_2, and $U_{x_2} = X \setminus \{x_1\}$ is is $Hpre$-open set in X containing x_2 but not x_1, therefore X is $Hpre - T_1$ space.

(2) Similar proof of (1).

Definition 5.22. Let (X,τ) be a topological space. Then the space (X,τ) is called $Hpre-T_2$ space iff for every distinct pair of points x_1 and x_2 in X, there exists $Hpre$-open sets U and V in Xs.t.$x_1 \in U$ and $x_2 \in V$, and $U \cap V = \emptyset$.
Theorem5.23.In any topological space $(X,\tau)$the statements are verified
(1) Every pre-T_2 is $Hpre-T_2$ space.
(2) Any$g$-T_2 is $Hpre-T_2$ space.
Proof. Direct that from Theorem 4.6.
Remark 5.24.(1)$Hpre-T_2$ space does not have to be pre-T_2 space (see Example 5.4(1)).
(2)$Hpre-T_2$space does not have to be g-T_2 space (see Example 3.9).
(3) Every $Hpre-T_2$ space is $Hpre-T_1$ space but the converse is not true(see Example 5.9(2)).
Theorem5.25. Let (X,τ)be T.S. and let (Y,σ) be T_2-space. Let $f: X \to Y$ be a one-one and $Hpre$-continuous function. Then (X,τ) is $Hpre-T_2$ space.
Proof. Let $x_1 \neq x_2 \in X$. Since f is one-one, $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$. Let $y_1 = f(x_1)$, $y_2 = f(x_2)$, so that $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then $y_1 \neq y_2 \in Y$. Since Y is T_2-space, then there exists a two open set U and V such that $y_1 \in U$ and $y_2 \in V$ and $U \cap V = \emptyset$. Since f is $Hpre$-continuous function, so $f^{-1}(U)$ and $f^{-1}(V)$ are $Hpre$-open set in X and $f^{-1}(U) \cap f^{-1}(V) = f(U \cap V)^{-1} = \emptyset$, $y_1 \in U$, $f^{-1}(y_1) \in f^{-1}(U)$ this implies $x_1 \in f^{-1}(U)$ and $y_2 \in V$, $f^{-1}(y_2) \in f^{-1}(V)$ this implies $x_2 \in f^{-1}(V)$. So (X,τ) is $Hpre-T_2$ space.
Theorem5.26. Let X be T_2-space (resp.pre-T_2 space) and $f: X \to Y Hpre$-open function, one-one and onto, then Y is $Hpre-T_2$ space.
Proof. Let $y_1 \neq y_2 \in X$. Since f is one-one and onto function, there exists $x_1 \neq x_2 \in X$ such that $y_1 = f(x_1)$, $y_2 = f(x_2)$, since X is T_2-space (resp.pre-T_2 space) then there exists are two open (resp. pre-open) sets U and V in X such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$, since f is $Hpre$-open function, therefore $f(U)$ and $f(V)$ are $Hpre$-open sets such that $y_1 = f(x_1) \in f(U)$, $y_2 = f(x_2) \in f(V)$ and $f(U) \cap f(V) = f(U \cap V) = \emptyset$. Then Y is $Hpre-T_2$ space.

ACKNOWLEDGEMENT

Mustafa Hasan and Luay A. Al-Swidi thanks the library in Facultyof Education for Pure Sciences in University of Bagdad for their hospitality during the preparation of this work,as well as the head of the Mathematics Departmentin University of Babylon to provide some references.

REFERENCES

[1] Jagtap, Abhijeet, Priya Malewadkar, Omkar Baswat, and Harshali Rambade. "Heart Disease Prediction using Machine Learning." International Journal of Research in Engineering, Science and Management 2, no. 2 (2019): p1-5.
[2] P. Agashe and N. Levine, “Adjacent topologies,” J. Math. Tokushima Univ, vol. 7 , pp. 21–35, 1973.
[3] D. Andrijevic, “Semi-preopen sets,” Mat. Vesn., vol. 38, no. 93 , pp. 24–32. 1986.
[4] K. Balachandran, P. Sundaram and J. Maki, “On generalized continuous maps in topological spaces,” Mem. Fac. Sci. Kochi Univ. Ser. A Math., vol. 12 , pp. 5–13, 1991.
[5] M. Caldas, T. Fukutake, S. Jafari and T. Noiri, “Some applications of delta-preopen sets in topological spaces,” Bull. Math. Acad. Sin., vol. 33, no. 3 , p. 261, 2005.
[6] M. Caldas and S. Jafari, “On g-US spaces,” Stud Cercet Stiint Ser Mat Univ Bacau, vol. 14 , pp. 13–20, 2005.
[7] H.M. Darwesh, “Between preopen and open sets in topological spaces,” Thai J. Math., vol. 11, no. 1 , pp. 143–155, 2012.
[8] W. Dunham, “A new closure operator for non-T-1 topologies,” Bull. Kyungpook Math. J., vol. 22 , pp. 55–60, 1982.
[9] W. Dunham, “T 1/2-spaces,” Kyungpook Math. J., vol. 17 , pp. 161–169, 1977.
[10] W. Dunham and N. Levine, “Further results on generalized closed sets in topology,” Kyungpook Math J ., vol. 20 , pp. 169–175, 1980.

[11] Y. B. Jun, S. W. Jeong, H. J. Lee and J. W. Lee, "Applications of pre-open sets," Appl. Gen. Topol., vol. 9, no. 2 , pp. 213–228, 2008.
[12] A. Kar and P. Bhattacharyya, "Some weak separation axioms," Bull. Calcutta Math. Soc., vol. 82 , pp. 415–422, 1990.
[13] N. Levine, "Generalized closed sets in topology," Rend. del Circ. Mat. di Palermo, vol. 19, no. 1, pp. 89–96, 1970.
[14] S. R. Malghan, "Generalized closed maps," J. Karnatak Univ. Sci., vol. 27, pp. 82–88, 1982.
[15] A. S. Mashour, M. E. A. El-Monsef and S. N. El-Deeb, "On precontinuous and weakprecontinuous functions," Proc. Math. Phys. Soc. Egypt , pp. 47–53, 1982.
[16] T. M. Nour, "Contributions to the theory of bitopological spaces,"
[17] Ph. D. Thesis, Delhi Univ. India, 1989.
[18] Md. Salamun Rashidin, Sara Javed, Bin Liu, Wang Jian, Suman Rajest S , "Insights: Rivals Collaboration on Belt and Road Initiatives and Indian Recourses" in Journal of Advanced Research in Dynamical and Control Systems, Vol. 11, SI. 04, PP. 1509-1522, 2019.
[19] Muliarta, I. N. Medical waste and its management at wangaya hospital in Denpasar. International Research Journal of Management, IT and Social Sciences, vol. 3, no. 5, pp. 94-102, 2016.
[20] H. Anandakumar, R. Arulmurugan, and C. C. Onn, Eds., "Computational Intelligence and Sustainable Systems," EAI/Springer Innovations in Communication and Computing, 2019.
[21] Ahmed, E. R., Islam, M. A., and Alabdullah, T. T. Y. The moderating role of Shariah supervisory board on sukuk pricing benchmark. International Journal of Excellence in Islamic Banking and Finance, Vol. 6, no. 2, pp. 1-32, 2008.
[22] D. Mukherjee and B. V. R. Reddy, "Design and development of a novel MOSFET structure for reduction of reverse bias pn junction leakage current," International Journal of Intelligence and Sustainable Computing, vol. 1, no. 1, p. 32, 2020.
[23] Dawood, S., Allawi, J. and Al-Khafaji, N. Employee's Attitude toward Electronic Administration Adoption: A Case Study of Al-Mustansiriyah University. International Journal of Advanced Engineering, Management and Science, vol. 2, no. 4, pp. 124-134, 2016.
[24] D. Datta, S. Mishra, and S. S. Rajest, "Quantification of tolerance limits of engineering system using uncertainty modeling for sustainable energy," International Journal of Intelligent Networks, vol. 1, pp. 1–8, 2020.