Binomial Distribution

The binomial distribution is a fundamental probability distribution in applied statistics, modeling situations where a random process or experiment has exactly two possible outcomes. It's applicable to n repeated trials, categorizing results as either success (p), the probability of the event happening, or failure (q), the probability of it not happening, with $p + q = 1$.The probability of achieving exactly x successes in n trials is given by:

$$P(x) = \binom{n}{x} p^x q^{n-x},$$

where: $\binom{n}{x} = \dfrac{n!}{x!\,(n-x)!}$.

Example 1: Following the shelling of the Popular Mobilization headquarters in Habbaniyah, residents of Anbar rushed to donate blood for the injured. Given that O^+ is the blood type in 60% of donations, what's the probability that exactly 2 of the 5 blood bags are O^+?

Solution: $p = 60\% = 0.6, q = 1 - 0.6 = 0.4$ and $n = 5, x = 2.$

$$P(2) = \binom{5}{2}(0.6)^2(0.4)^3 = \frac{5!}{2!\,3!} \times 0.36 \times 0.064 = 0.2304.$$

Example 2: In a population where 10% are colorblind, find the probabilities for a random sample of 25 people:

(a) Exactly 3 are colorblind. (b) At most 2 are colorblind.
(c)At least 2 are colorblind. (d) 2, 3, or 4 are colorblind.

Solution: $p = 10\% = 0.1, q = 1 - 0.1 = 0.9$ and $n = 25.$

$$(a)\ x = 3,\quad P(3) = \binom{25}{3}(0.1)^3(0.9)^{22} = \frac{25!}{3!\,22!} \times 0.001 \times 0.0985$$

$$= \frac{25 \times 24 \times 23 \times 22!}{3 \times 2 \times 22!} \times 0.001 \times 0.0985 = 0.2265$$

$$(b)\ x \le 2 \Rightarrow\ x = 0 \text{ or } 1 \text{ or } 2 \quad \Rightarrow\ P(x \le 2) = P(0) + P(1) + P(2)$$

$$P(x \le 2) = \binom{25}{0}(0.1)^0(0.9)^{25} + \binom{25}{1}(0.1)^1(0.9)^{24} + \binom{25}{2}(0.1)^2(0.9)^{23}$$

$$= 0.0718 + 0.1995 + 0.2658 = 0.5371$$

$$(c)\ P(x \ge 2) = 1 - P(x \le 1) = 1 - \big(P(0) + P(1)\big)$$

$$= 1 - (0.0718 + 0.1995) = 0.7287$$

$$(d)\ P(2) + P(3) + P(4) = 0.2658 + 0.2265 + \binom{25}{4}(0.1)^4(0.9)^{21} = 0.6307$$

In the context of a binomial distribution, three important measures are:

- The mean (μ), which represents the expected number of successes in n trials, is calculated as $\boxed{\mu = np}$, where n is the number of trials and p is the probability of success.

- The variance (σ^2), indicating the spread of the distribution, is given by $\boxed{\sigma^2 = npq}$, with q being the probability of failure $(1 - p)$.

- The standard deviation (σ), a measure of the amount of dispersion, is the square root of the variance, $\boxed{\sigma = \sqrt{npq}.}$

Example 3: Only 1.5% of people worldwide have green eyes. What's the coefficient of variation for green eye color in a random group of 20 people?

Solution: $p = 0.015, q = 1 - 0.015 = 0.985, n = 20$

$$\mu = np = 20 \times 0.015 = 0.3$$

$$\sigma^2 = npq = 0.3 \times 0.985 = 0.2955$$

$$C.V = \frac{\sigma}{\mu} \times 100\% = \frac{\sqrt{0.2955}}{0.3} \times 100\% = 181.2\%$$

Example 4: A binomial variable x has a mean of 2 and variance of 1.5. What's the probability of x is less than 3?

Solution: $\mu = np = 2 \qquad (1)$

$$\sigma^2 = npq = 1.5 \qquad (2)$$

$$(2) \div (1) \Rightarrow \frac{npq}{np} = \frac{1.5}{2} \Rightarrow q = 0.75$$

$$p = 1 - q = 1 - 0.75 = 0.25$$

$$np = 2 \Rightarrow 0.25n = 2 \Rightarrow n = 8$$

$$P(x < 3) = P(0) + P(1) + P(2)$$

$$= \binom{8}{0}(0.25)^0(0.75)^8 + \binom{8}{1}(0.25)^1(0.75)^7 + \binom{8}{2}(0.25)^2(0.75)^6$$

$$= 0.1001 + 0.2670 + 0.3115 = 0.6786$$

Example 5: A binomial variable x has $n = 4$. If $P(2) = P(3)$, find the probability of x at most 2.

Solution:

$$P(2) = P(3) \;\Rightarrow\; \binom{4}{2} p^2 q^{4-2} = \binom{4}{3} p^3 q^{4-3}$$

$$6\, p^2 q^2 = 4p^3 q \;\Rightarrow\; 3q = 2p$$

$$3(1-p) = 2p \;\Rightarrow\; 2p + 3p = 3 \;\Rightarrow\; p = 0.6 \text{ and } q = 0.4$$

$$P(x \le 2) = P(0) + P(1) + P(2)$$

$$P(x \le 2) = \binom{4}{0}(0.6)^0(0.4)^4 + \binom{4}{1}(0.6)^1(0.4)^3 + \binom{4}{2}(0.6)^2(0.4)^2$$

$$= 0.0256 + 0.1536 + 0.3456 = 0.5248$$

H.W.

1. Given that about 10% of the population is left-handed, find the probabilities for a random sample of 25 people:

(a) Exactly 2 are left-handed. (b) At most 3 are left-handed.

(c) At least 3 are left-handed. (d) 3, 4, or 5 are left-handed.

2. In a certain city, 8% of the residents have a rare blood type. What's the coefficient of variation for this blood type in a random sample of 15 residents?

3. A binomial variable x has a mean of 4 and variance of 3. What's the probability of x is less than 2?

4. A binomial variable x has $n = 5$. If $P(1) = P(2)$, find the probability of x at least 2.