- Schrödinger's Equations:

1- Time – dependent Schrödinger equation (TDSE):

A free particle moving along x- axis with a defined momentum is described by the an infinite plane wave

$\Psi\,(x,t)$=A $e^{i(K_x\,x-\omega t)}$ .... (1)

$K_x = \frac{p_x}{\hbar}$ and $\omega = \frac{E}{\hbar}$ .... (2)

$K = \frac{2\pi}{\lambda} = \frac{2\pi}{h/p} = \frac{2\pi\, p}{h} = \frac{p}{\hbar}$

$\omega = 2\,\pi\nu$, $E = h\,\nu$, $\nu = \frac{E}{h}$

$\omega = 2\,\pi\frac{E}{h} = \frac{E}{\hbar}$

Substituting Eqn. (2) in (1) we get:

$\Psi\,(x,t)$=A $e^{\frac{i}{\hbar}(p_x\,x-Et)}$ .... (3)

Differentiation of Eqn. (3) with respect to time gives:

$\frac{\partial\Psi}{\partial t} = \mathrm{A}e^{\frac{i}{\hbar}(p_x\,x-Et)}.\left(-\frac{i}{\hbar}\,E\right)$

$\frac{\partial\Psi}{\partial t} = \left(-\frac{i}{\hbar}\,E\right)\Psi\,(x,t)$

$i\hbar$ بالضرب في

$i\hbar\,\frac{\partial\Psi}{\partial t} = E\,\Psi\,(x,t)$ .... (4)

Successive differentiation of Eqn. (3) with respect to x leads to:

$\frac{\partial\Psi}{\partial x} = \mathrm{A}e^{\frac{i}{\hbar}(p_x\,x-Et)}.\left(\frac{i}{\hbar}\,p_x\right)$

$\frac{\partial \Psi}{\partial x} = \frac{i}{\hbar}\, p_x\ \Psi\,(x,t)$

$-i\hbar$ بالضرب في

$-i\hbar\,\frac{\partial \Psi}{\partial x} = p_x\ \Psi\,(x,t)$ ……….…….5

$\frac{\partial^2 \Psi}{\partial x^2} = \mathrm{A}e^{\frac{i}{\hbar}(p_x x - Et)} \cdot \left(\frac{i}{\hbar}\, p_x\right) \cdot \left(\frac{i}{\hbar}\, p_x\right)$

$\frac{\partial^2 \Psi}{\partial x^2} = -\,\frac{{p_x}^2}{\hbar^2}\,\Psi\,(x,t)$

$-\frac{\hbar^2}{2m}$ بضرب الطرفين بالمقدار

$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} = \frac{{p_x}^2}{2m}\,\Psi\,(x,t)$ …………….6

Classically for a free nonrelativistic particle

$E = \frac{{p_x}^2}{2m} + V\,(x,t)$ …………….7

Hence from Eqns. (4) and (7) we may write:

$i\hbar\,\frac{\partial \Psi}{\partial t} = E\,\Psi\,(x,t) = \left[\frac{{p_x}^2}{2m} + V\,(x,t)\right]\Psi\,(x,t)$

$= \frac{{p_x}^2}{2m}\,\Psi\,(x,t) + V\,(x,t)\Psi\,(x,t)$

Substituting Eqn. (6) in above Eqn. we get:

$$\therefore\ \boxed{i\hbar\,\frac{\partial \Psi}{\partial t} = -\,\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\,(x,t)\Psi\,(x,t)}$$ ……………8

Eqn. (8) is the **1-dimensional time-dependent Schrödinger equation**. For a free particle, V(x, t) =0

In 3- dimensions Eqn. (8) becomes:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(r,t)\, \Psi(r,t) \quad \text{……………..9}$$

Where $\nabla^2 = \frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} + \frac{\partial^2}{\partial^2 z}$

The operator for the Hamiltonian (H) of the system is:

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(r,t)$$

Hence Eqn. (9) can be written in the form:

$$i\hbar \frac{\partial \Psi}{\partial t} = H\ \Psi(r,t) \quad \text{………………10}$$

So eqns. (4) and (5) written in the form

$$\left(i\hbar \frac{\partial}{\partial t}\right) \Psi = E\ \Psi \qquad \therefore\ E = i\hbar \frac{\partial}{\partial t}$$

$$\left(-i\hbar \frac{\partial}{\partial x}\right) \Psi = p_x\ \Psi \qquad \therefore\ p_x = -i\hbar \frac{\partial}{\partial x}$$

In 3 – dimensions

$E = i\hbar \frac{\partial}{\partial t}$ and $p = -i\hbar\, \nabla$