Multiple Integrals

The multiple integral is a mathematical concept that extends the notion of a definite integral to functions of multiple real variables. For instance, integrals of a function of two variables are commonly referred to as double integrals, whereas those of a function of three variables are termed triple integrals, and so on.

Double Integrals

The expression $\int_c^d \int_a^b f(x,y)dxdy$ is called double integral, indicating that we first integrate with respect to x (holding y constant) from x = a to x = b, and subsequently integrate the outcome with respect to y from y = c to y = d.

Example 1: Evaluate $\int_1^2 \int_2^4 (x+2y)dxdy$

$$\int_1^2 \int_2^4 (x+2y)dxdy = \int_1^2 \frac{x^2}{2} + 2yx\Big|_2^4 dy$$

$$= \int_1^2 (8 + 8y - 2 - 4y)\, dy$$

$$= \int_1^2 (4y+6)\, dy = 2y^2 + 6y\Big|_1^2$$

$$= 8 + 12 - 2 - 6 = 12$$

An alternative approach is to evaluate the inner integral first, obtaining its result, and then proceed to evaluate the outer integral.

$$\int_2^4 (x+2y)dx = \frac{x^2}{2} + 2yx\Big|_2^4 = 4y + 6$$

$$\int_1^2 \int_2^4 (x+2y)dxdy = \int_1^2 (4y+6)dy = 12$$

Example 2: Evaluate $\int_0^1\int_y^1 2xy\,dxdy$

$$\int_0^1\int_y^1 2xy\,dxdy = \int_0^1 x^2\Big|_y^1 ydy$$

$$= \int_0^1 (1-y^2)ydy = \int_0^1 (y-y^3)dy$$

$$= \frac{y^2}{2} - \frac{y^4}{4}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Example 3: Evaluate $\int_0^\pi\int_0^{\sin x} y\,dydx$

$$\int_0^\pi\int_0^{\sin x} y\,dydx = \int_0^\pi \frac{y^2}{2}\Big|_0^{\sin x} dx = \int_0^\pi \frac{\sin^2 x}{2} dx = \int_0^\pi \frac{1-\cos 2x}{4} dx$$

$$= \frac{1}{4}x - \frac{1}{8}\sin 2x\Big|_0^\pi = \frac{\pi}{4}$$

Example 4: Evaluate $\int_1^2\int_y^{y^2} dxdy$

$$\int_1^2\int_y^{y^2} dxdy = \int_1^2 x\Big|_y^{y^2} dy = \int_1^2 (y^2-y)\,dy = \frac{y^3}{3} - \frac{y^2}{2}\Big|_1^2 = \frac{5}{6}$$

Example 5: Evaluate $\int_0^1\int_0^x \sqrt{1-x^2}\,dydx$

$$\int_0^1\int_0^x \sqrt{1-x^2}\,dydx = \int_0^1 y\Big|_0^x \sqrt{1-x^2}dx = \int_0^1 x\sqrt{1-x^2}\,dx = \frac{-1}{2}\int_0^1 -2x(1-x^2)^{1/2}\,dx$$

$$= -\frac{1}{2}(1-x^2)^{3/2} \times \frac{2}{3}\Big|_0^1 = -\frac{1}{3}(0-1) = \frac{1}{3}$$

Triple Integrals

A Triple Integral is a multi-variable integration type involving three variables, typically representing volume, which is why it's also referred to as Volume Integral. The process of evaluating such an integral is called Triple Integration.

Let's work through a simple example to illustrate this concept.

Example 6: Evaluate $\int_0^2\int_{-1}^1\int_0^1 (2x-y+z)dxdydz$

$$\int_0^2\int_{-1}^1\int_0^1 (2x-y+z)dxdydz = \int_0^2\int_{-1}^1 (x^2-yx+zx)\Big|_0^1 dydz = \int_0^2\int_{-1}^1 (1-y+z)dydz$$

$$= \int_0^2 \left(y-\frac{y^2}{2}+zy\right)\Bigg|_{-1}^1 dz = \int_0^2 \left(1-\frac{1}{2}+z-\left(-1-\frac{1}{2}-z\right)\right)dz$$

$$= \int_0^2 (2+2z)dz = 2z+z^2 \Big|_0^2 = 8$$

Example 7: Evaluate $\int_1^5\int_{-2}^1\int_0^3 (x^2yz)dydxdz$

$$\int_1^5\int_{-2}^1\int_0^3 (x^2yz)dydxdz = \int_1^5\int_{-2}^1 \left(x^2\frac{y^2}{2}z\right)\Bigg|_0^3 dxdz$$

$$= \int_1^5\int_{-2}^1 \left(\frac{9}{2}x^2z\right)dxdz = \int_1^5 \frac{3x^3}{2}\Bigg|_{-2}^1 zdz$$

$$= \int_1^5 \left(\frac{3}{2}-\frac{3(-2)^3}{2}\right)zdz = \int_1^5 \frac{27}{2}zdz$$

$$= \frac{27}{4}z^2\Bigg|_1^5 = 162$$

Example 8: Evaluate $\int_0^\pi\int_0^\pi\int_0^3 x^2\sin\theta\,dxd\theta d\emptyset$

$$\int_0^\pi\int_0^\pi\int_0^3 x^2\sin\theta\,dxd\theta d\emptyset=\int_0^\pi\int_0^\pi\left(\frac{x^3}{3}\Big|_0^3\right)\sin\theta\,d\theta d\emptyset=\int_0^\pi\int_0^\pi 9\sin\theta\,d\theta d\emptyset$$

$$=\int_0^\pi -9\cos\theta\Big|_0^\pi d\emptyset=\int_0^\pi -9(-1-1)d\emptyset=\int_0^\pi 18d\emptyset=18\emptyset\Big|_0^\pi=18\pi$$

H.W.

Evaluate each of the following integrals

1. $\int_0^2\int_1^{e^x} dydx$

2. $\int_0^1\int_{\sqrt{y}}^1 dxdy$

3. $\int_{-2}^1\int_{x^2+4x}^{3x+2} dydx$

4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xydxdy$

5. $\int_0^1\int_{x^2}^1 (x+y)dydx$

6. $\int_0^1\int_0^{y^2} \sqrt{y^3+3}\,dxdy$

7. $\int_0^1\int_0^2 x\sqrt{4-x^2}\,dxdy$

8. $\int_0^2\int_1^3\int_1^2 xy^2zdxdydz$

9. $\int_0^2\int_0^1\int_0^1 (x^2+y^2+z^2)dxdydz$

10. $\int_0^2\int_{3x/2}^3\int_0^{5x/2} dzdydx$