First-Order Differential Equations

The general first-order differential equation is written as:

$$\frac{dy}{dx} = F(x,y) \quad \text{or} \quad M(x,y)dx + N(x,y)dy = 0$$

Separable Differential Equations: A first-order ODE is separable if it can be written in the form $f(x)dx = g(y)dy$ where the function $f(x)$ is independent of y and $g(y)$ is independent of x. We can find the general solution of this differential equation by integral

$$\int f(x)dx = \int g(y)dy$$

Example 1: Solve the ODE $xdy - 3ydx = 0$.

Solution: By separating the variables it becomes $\dfrac{dy}{y} = \dfrac{3dx}{x}$

$$\int \frac{dy}{y} = 3\int \frac{dx}{x} \Rightarrow \quad \ln y = 3\ln x + \ln C \quad \Rightarrow \quad \ln y = \ln x^3 + \ln C$$

$$\ln y = \ln Cx^3 \qquad \Rightarrow \qquad y = Cx^3$$

Example 2: Find the general solution of the ODE $y' + 2e^{2x}y = e^{2x}y^2$.

Solution: From the given equation, we have $y' = (y^2 - 2y)e^{2x}$

$$\frac{dy}{dx} = (y^2 - 2y)e^{2x} \quad \Rightarrow \quad \frac{1}{y(y-2)}dy = e^{2x}dx$$

$$\frac{1}{y(y-2)} = \frac{A}{y} + \frac{B}{y-2}; \quad y = 0 \Rightarrow A = -\frac{1}{2} \quad , \quad y = 2 \Rightarrow B = \frac{1}{2}$$

$$\int e^{2x}dx = \int \left[\frac{1}{2(y-2)} - \frac{1}{2y}\right] dy$$

$$\frac{1}{2}e^{2x} + C_1 = \frac{1}{2}\ln(y-2) - \frac{1}{2}\ln y$$

$$e^{2x} + 2C_1 = \ln(y-2) - \ln y \quad \Rightarrow \quad e^{2x} + C = \ln\left(\frac{y-2}{y}\right)$$

Example 3: Solve $\dfrac{dQ}{dx} + \dfrac{1}{2}Q = \dfrac{5}{2}$ with $Q(1) = 6$

Solution: We change to differential form, separate the variables, and integrate

$$\frac{dQ}{dx} = \frac{5}{2} - \frac{1}{2}Q \quad \Rightarrow \quad \frac{dQ}{dx} = -\frac{Q-5}{2}$$

$$\int_6^Q \frac{dy}{y-5} = -\frac{1}{2}\int_1^x dt$$

$$\ln(y-5) \Big|_6^Q = -\frac{1}{2}t\Big|_1^x$$

$$\ln(Q-5) - \ln 1 = -\frac{1}{2}x + \frac{1}{2} \quad \Rightarrow \quad \ln(Q-5) = \frac{1-x}{2}$$

$$Q - 5 = e^{(1-x)/2} \quad \Rightarrow \quad Q = 5 + e^{(1-x)/2}$$

Example 4: Solve $e^x \tan y\, dx = (1+e^x)\sec^2 y\, dy$ with $y(\ln 2) = \pi/4$

Solution:

$$\int_{\pi/4}^{y} \frac{\sec^2 u}{\tan u} du = \int_{\ln 2}^{x} \frac{e^t}{1+e^t} dt \quad \Rightarrow \quad \ln(\tan u) \Big|_{\pi/4}^{y} = \ln(1+e^t)\Big|_{\ln 2}^{x}$$

$$\ln(\tan y) - \ln(1) = \ln(1+e^x) - \ln\left(1+e^{\ln 2}\right)$$

$$\ln(\tan y) = \ln(1+e^x) - \ln(3)$$

$$\ln(\tan y) = \ln\frac{1+e^x}{3}$$

$$\tan y = \frac{1+e^x}{3}$$

$$y = \tan^{-1}\left(\frac{1+e^x}{3}\right)$$

Newton's Low of Cooling: The rate of loss of heat from a body is directly proportional to the difference in the temperature of the body and its ambient.

$$Newton's\ law\ of\ cooling\ is\ given\ by: \quad \frac{dT}{dt} = -k(T - M)$$

where t is time, k is the constant of proportionality, and M is the ambient temperature.

Example 5: Placed a metal bar, at a temperature of $40°C$ in a room with constant temperature of $0°C$. After 20 minutes the temperature of the bar is $30°C$. Determine the time required to reach the bar at a temperature of $20°C$.

Solution: $M = 0°F \quad ⇨ \quad \frac{dT}{dt} = -kT \quad ⇨ \quad \frac{dT}{T} = -kdt \quad ⇨ \quad \ln T = -kt + c_1$

$$T = e^{-kt+c_1} = e^{-kt}.e^{c_1}$$

$$\boxed{T = ce^{-kt}} \quad ; (e^{c_1} = \mathrm{c})$$

$(Initial\ condition) \quad t = 0, \quad T = 40°C$

$$40 = ce^{-k\times 0} \quad ⇨ \quad c = 40$$

$$T = 40e^{-kt}$$

To find k we have $t = 20\ min$, $T = 30°C \quad ⇨ \quad 30 = 40e^{-k\times 20}$

$e^{-20k} = 0.75 \quad ⇨ \quad -20k = \ln(0.75)$

$$\therefore \quad k = \frac{\ln(0.75)}{-20} = 0.0144$$

$$T = 40e^{-0.0144t}$$

$$20 = 40e^{-0.0144t} \ ⇨ \ e^{-0.0144t} = 0.5 \ ⇨ \ t = \frac{\ln(0.5)}{-0.0144} = 48 \text{ minutes.}$$

Example 6: Water at a temperature of $80°C$ is placed in a room which is held at a constant temperature of $25°C$. How much would be the temperature of water after 10 minutes if $k = 0.056$.

Solution: $\frac{dT}{dt} = -0.056(T - 25) \ ⇨ \ \frac{dT}{(T - 25\ )} = -0.056dt$

$$\ln(T - 25) = -0.056t + c_1 \quad ⇨ \quad T - 25 = e^{-0.056t+c_1}$$

$$T = 25 + ce^{-0.056t} \quad ; c = e^{c_1}$$

$$t = 0 \quad , \quad T = 80 \ ⇨ \ 80 = 25 + c \ ⇨ \ c = 55$$

$$T = 25 + 55e^{-0.056t} \ ⇨ \ T(10) = 25 + 55e^{-0.056\times 10} = 56.4°C$$

Radioactive Decay: Radioactive decay is the random process in which a nucleus loses energy by emitting radiation. The number of decays per second is given by:

$$\frac{dN}{dt} = \lambda N$$

where λ is the decay constant and N represent to the number of undecayed nuclei.
The half-life $T_{1/2}$ of a sample is the time taken for half of the radioactive nuclei to decay:

$$T_{1/2} = -\frac{\ln 2}{\lambda}$$

Example 7: The half-life of radioactive radium is 1600 years. If a sample initially contains 50 gm, how long will it be until it contains 45 gm?

Solution:

$$T_{1/2} = -\frac{\ln 2}{\lambda} \quad ⇰ \quad 1600 = -\frac{\ln 2}{\lambda} \quad ⇰ \quad \lambda = -\frac{\ln 2}{1600}$$

$$\frac{dN}{dt} = \lambda N \quad ⇰ \quad \frac{dN}{N} = \lambda dt$$

$$\int \frac{dN}{N} = \int \lambda dt \quad ⇰ \quad \ln N = \lambda t + C$$

$$N = Ae^{\lambda t}; A = e^{C}$$

We have $N = 50$ when $t = 0$ ⇰ $50 = Ae^{0}$ ⇰ $A = 50$

Thus $N = 50e^{\lambda t}$

So, $N = 50e^{-\frac{\ln 2}{1600}t}$ ⇰ $\frac{N}{50} = e^{-\frac{\ln 2}{1600}t}$

$$\frac{45}{50} = e^{-\frac{\ln 2}{1600}t} \quad ⇰ \quad -\frac{\ln 2}{1600} t = \ln\left(\frac{45}{50}\right)$$

$$t = -\frac{1600 \ln(0.9)}{\ln 2} = 243.2 \quad \text{years}$$

H.W

Solve the ODEs:

(1) $x(2y-3)dx + (x^2+1)dy = 0$ (Ans. $(x^2+1)(2y-3) = K$)

(2) $t^2dT + tTdt = (t+6)dT + 2Tdt.$ (Ans. $T^5(t+2)^4(t-3) = K$)

(3) $(1+u^2)dv - (1+v^2)du = 0$ with $v(0) = 1$

(4) $\dfrac{ds}{dt} = \dfrac{s^2 - s - 2}{t^2 + t}$ with $s(1) = 3$

(5) A body at a temperature of $40°C$ is placed in a room with constant temperature of $20°C$. If after 10 minutes the temperature of the body is $35°C$, find the time required for the body to reach a temperature of $30°C$. (Ans: 14.096 minutes)

(6) The half-life of radioactive einsteinium is 276 days. After 100 days, 0.5 gram remains. What was the initial amount?

(7) The half-life of radioactive radium is 1600 years. If a sample initially contains 30 grams, how much the amount will remain after 250 years? (Ans: 26.9 grams)

Websites:

1. https://tutorial.math.lamar.edu/Classes/DE/Separable.aspx
2. https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/sec_first_order_homogeneous_linear.html