6- Velocity Transformations:

A gain considers two inertial systems S and Ś moving with relative velocity v along the $X\acute{X}$ axes. Consider a particle at P. The velocity of this particle in S is u and $\acute{u}$ in Ś .

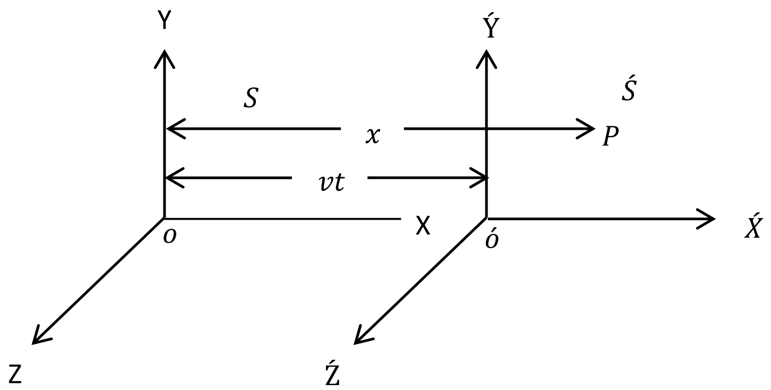
The velocity component in S and Ś are

$$u_x = \frac{dx}{dt}\ ,\ u_y = \frac{dy}{dt}\ ,\ u_z = \frac{dz}{dt} \text{ ---------------- (1)}$$

and

$$u_{\acute{x}} = \frac{d\acute{x}}{d\acute{t}}\ ,\ u_{\acute{y}} = \frac{d\acute{y}}{d\acute{t}}\ ,\ u_{\acute{z}} = \frac{d\acute{z}}{d\acute{t}} \text{ --------------- (2)}$$

Differentiation of the Lorentz Eqn. gives:



$$d\acute{x} = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}$$

$$d\acute{y} = dy$$

$$d\acute{z} = dz$$

$$d\acute{t} = \frac{dt - \frac{vdx}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Substitution of these values in Eqn. (2) gives:

$$u_{\acute{x}} = \frac{dx - vdt/\sqrt{1 - v^2/c^2}}{dt - \frac{vdx}{c^2}/\sqrt{1 - v^2/c^2}} = \frac{dx - vdt}{dt - \frac{vdx}{c^2}}$$

Divided by dt we get:

$$u_{\acute{x}} = \frac{\left(\frac{dx}{dt}\right) - v}{1 - \frac{v}{c^2}\left(\frac{dx}{dt}\right)}$$

Substitute Eqn. (1) we get

$$u_{\acute{x}} = \frac{u_x - v}{1 - \frac{v}{c^2}u_x}$$

Similarly $$u_{\acute{y}} = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c^2}u_x}$$

$$u_{\acute{z}} = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - \frac{v}{c^2}u_x}$$

These are the Lorentz velocity transformations. The invers transformation is obtained by replacing v by $-v$ and interchanging primed and unprimed coordinates:

$$u_x = \frac{u_{\acute{x}} + v}{1 + \frac{v}{c^2} u_{\acute{x}}}$$

$$u_y = \frac{u_{\acute{y}} \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u_{\acute{x}}}$$

$$u_z = \frac{u_{\acute{z}} \sqrt{1 - v^2/c^2}}{1 + \frac{v}{c^2} u_{\acute{x}}}$$

Example: Imagining that light emitted along the $\acute{x}$ – axis, if $u_{\acute{x}} = c$ with respect to (w.r.t.) $\acute{S}$. Find the velocity of light w.r.t. S.

Solution:

$$\because u_x = \frac{u_{\acute{x}}+v}{1+\frac{v}{c^2}u_{\acute{x}}} \qquad \text{and since } u_{\acute{x}} = c$$

$$\therefore u_x = \frac{c + v}{1 + \frac{v}{c^2} c} = \frac{c + v}{1 + \frac{v}{c}} = \frac{c(c + v)}{(c + v)}$$

$\therefore u_x = c$ اي ان سرعة الضوء في كلا المرجعين لها نفس القيمة

7- Length Contraction:

The first of the interesting consequences of the Lorentz Transformation is that length no longer has an absolute meaning: the length of an object depends on its motion relative to the frame of reference in which its length is being measured. Let us consider a rod moving with a velocity vx relative to a frame of reference S, and lying along the X axis. This rod is then stationary relative to a frame of reference $\acute{S}$ which is also moving with a velocity vx relative to S.

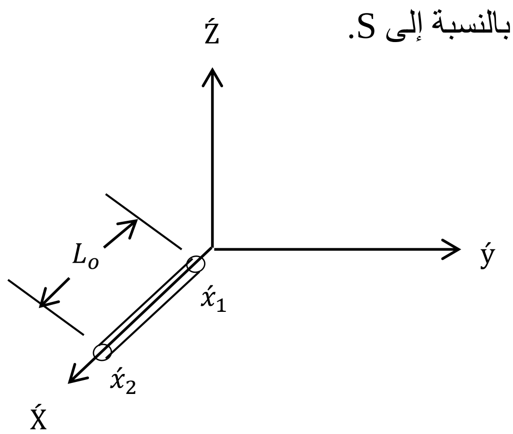
أول النتائج المثيرة للاهتمام لتحويل Lorentz هو أن الطول لم يعد له معنى مطلق: يعتمد طول الجسيم على حركته بالنسبة للإطار المرجعي الذي يتم قياس طوله فيه. لنعتبر قضيب يتحرك بسرعة vx بالنسبة لإطار مرجعي S ، ويقع على طول المحور X. ثم يكون هذا القضيب ثابتًا بالنسبة للإطار المرجعي Ś والذي يتحرك أيضًا بسرعة vx بالنسبة إلى S.

The length of the rod, as measured in Ś is then

$$L_o = \acute{x}_2 - \acute{x}_1$$

Where L_o is known as the *proper length* of the rod.

To an observer in S, the length of the rod is:



$$L = x_2 - x_1$$

Using Lorentz transformation

$$\acute{x}_1 = \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}$$

$$\acute{x} = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}}$$

Substituting these expression in L_o

$$L_o = \frac{x_2 - vt_2}{\sqrt{1 - v^2/c^2}} - \frac{x_1 - vt_1}{\sqrt{1 - v^2/c^2}}$$

$$L_o = \frac{x_2 - vt - x_1 - vt}{\sqrt{1 - v^2/c^2}} = \frac{x_2 - x_1}{\sqrt{1 - v^2/c^2}}$$

$$L_o = \frac{L}{\sqrt{1 - v^2/c^2}}$$

8- Time Dilation:

Perhaps the most unexpected consequence of the Lorentz transformation is the way in which our ‘commonsense’ concept of time has to be drastically modified.

ربما تكون النتيجة غير المتوقعة لتحويل لورنتز هي الطريقة التي يجب بها تعديل مفهومنا "المنطقي" للوقت بشكل جذري.

Consider two successive events occurring at the same point $\acute{x}$ in the interval frame $\acute{S}$. Let $\acute{t}_1$ and $\acute{t}_2$ be the times recorder by an observer in frame $\acute{S}$. Then time interval measured by the observer is $\acute{t}_1$ - $\acute{t}_2$. For the observer in frame S the time measured is $\Delta t = t_2 - t_1$. The time interval between events in $\acute{S}$ is called the Proper time $\Delta\tau = \acute{t}_2$ - $\acute{t}_1$

$$t_1 = \frac{\acute{t}_1 + \frac{v\acute{x}}{c^2}}{\sqrt{1 - v^2/c^2}}$$

$$t_2 = \frac{\acute{t}_2 + \frac{v\acute{x}}{c^2}}{\sqrt{1 - v^2/c^2}}$$

Then the time dilation is

$$\Delta t = t_2 - t_1 = \frac{\acute{t}_2 + \frac{v\acute{x}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{\acute{t}_1 + \frac{v\acute{x}}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\acute{t}_2 - \acute{t}_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{\Delta\tau}{\sqrt{1 - v^2/c^2}}$$

Example 1: At what speed does the relativistic value for length differ from the classical value by (1) percent?

Solution:

$$\frac{L_o - L}{L_o} = 0.01$$

or $L_o - L = 0.01\, L_o$

$\therefore L = L_o - 0.01\, L_o = 0.99 L_o \quad \Rightarrow \frac{L}{L_o} = 0.99$ ----------------- (1)

Since $L = L_o\sqrt{1 - v^2/c^2} \quad \Rightarrow \frac{L}{L_o} = \sqrt{1 - v^2/c^2}$ ---------------- (2)

بمساواة المعادلتين (1) و (2) نحصل على

$$\sqrt{1 - v^2/c^2} = 0.99 \quad \Rightarrow \quad 1 - \frac{v^2}{c^2} = 0.98$$

$$\frac{v^2}{c^2} = 1 - 0.98 = 0.02 \quad \Rightarrow \quad v^2 = 0.02c^2$$

Therefore $v = \sqrt{0.02c^2} = 141c$

Example 2: A beam of particles travels at a speed of (0.9c). At this speed, the mean lifetime as measured in the laboratory frame is ($5\text{x}10^{-6}$) . What is the proper lifetime of the particles?

Solution: proper lifetime $\Delta\tau = \Delta t\sqrt{1 - v^2/c^2}$

$$\Delta\tau = (5 \times 10^{-6})\sqrt{1 - (0.9)^2} = 2.18 \times 10^{-6}\ sec.$$

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9 - Relativistic Mass:

In Newtonian mechanics, mass is considered to be a constant quantity independent of its velocity. In general, if a mass is moving with a velocity (v) relative to an observer then $m = \frac{m_o}{\sqrt{1 - v^2/c^2}}$

Where m is the effective mass and m_0 is the rest mass.

10 - Relativistic Momentum:

Any relativistic generalization of Newtonian momentum must satisfy two criteria:

أي تعميم نسبي للزخم النيوتيوني يجب أن يفي بمعيارين

1. Relativistic momentum must be conserved in all frames of reference.

الزخم النسبي يجب ان بكون محفوظ في جميع الأطر المرجعية.

2. Relativistic momentum must reduce to Newtonian momentum at low speeds.

يجب أن يقلل الزخم النسبي إلى الزخم النيوتيوني عند السرعات المنخفضة

The first criterion must be satisfied in order to satisfy Einstein's first postulate, while the second criterion must be satisfied as it is known that Newton's Laws are correct at sufficiently low speeds.

المعيار الأول يجب ان يحقق فرضية أينشتاين الأولى ، بينما المعيار الثاني يجب أن يحقق قوانين نيوتن التي تكون صحيحة عند السرعات المنخفضة بدرجة كافية.

The relativistic momentum of a particle moving with a velocity (v) as measured with respect to a frame of reference S, that satisfies these criteria can be shown to take the form:

$$p = mv = \frac{m_o v}{\sqrt{1-v^2/c^2}}$$

Where m_o is the rest mass of the particle, i.e. the mass of the particle when at rest, and which can be identified with the Newtonian mass of the particle. Einstein then postulated that, for a system of particles:

The total momentum of a system of particles is always conserved in all frames of reference, whether or not the total number of particles involved is constant.

الزخم الكلي لنظام مكون من الجسيمات يكون دائما محفوظ في جميع الإطارات من المرجع ، سواء كان العدد الكلي للجسيمات المعنية ثابتًا أم لا

11 - Total Relativistic Energy:

We can now define a quantity E by: $T = mc^2 - m_o c^2$

or $E = T + m_o c^2$

This relation implies that mass is a form of energy. Einstein called $m_o c^2$, the rest energy of object . By analogy, mc^2 the sum of kinetic energy and rest energy is called the total energy E. That is $E = mc^2$

The rest energy of an electron is

$$E_o = m_o c^2 = 9.11 * 10^{-31} kg \times (3 * 10^8 \; m/sec)^2 = 8.2 * 10^{-14} J$$

$$\therefore E_o = \frac{8.2*10^{-14} J}{1.6*10^{-19} J/eV} = 0.511 * 10^6 eV = 0.511 \; MeV$$

Often, mass is expressed in terms of MeV/c^2 so that the electron mass is $0.511 MeV/c^2$.

- A useful relation connect the total energy E , momentum p , and rest energy $E_o = m_o c^2$

$$E = \sqrt{p^2 c^2 + m_o^2 c^4}$$

We can obtain as follow:

We have $m = \frac{m_o}{\sqrt{1-v^2/c^2}}$

Multiplying both side of this expression by c^2 , and squaring we get:

$$mc^2 = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} \quad \Rightarrow \quad m^2 c^4 \left(1 - \frac{v^2}{c^2} \right) = m_o^2 c^4$$

$$m^2 c^4 = m^2 v^2 c^2 + m_o^2 c^4$$

Using $p = mv$, and $E = mc^2$ we get $E^2 = p^2 c^2 + m_o^2 c^4 \quad \Rightarrow$

$$E = \sqrt{p^2 c^2 + m_o^2 c^4}$$