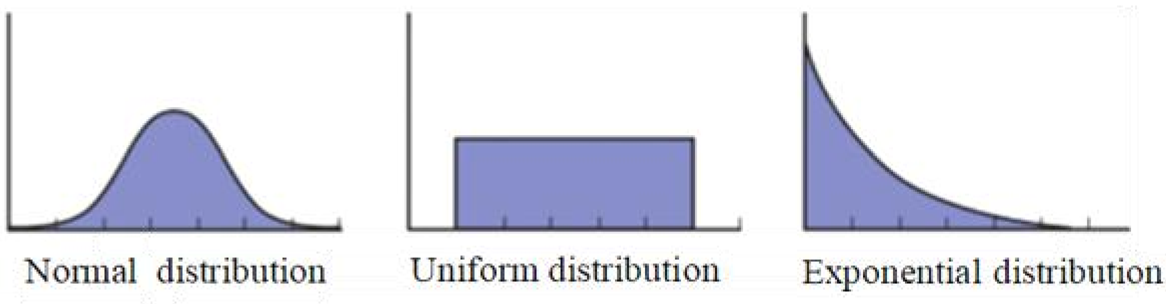
Continuous Probability Distributions

Let's explore the distributions of continuous random variables. A continuous variable is characterized by its ability to take on any value within a given range, implying that between any two values, there exists an infinite number of possible values.

A function $f(x)$ is called a probability distribution (probability density function) of the continuous random variable x if the total area bounded by its curve and the x −axis is equal to 1. That is:

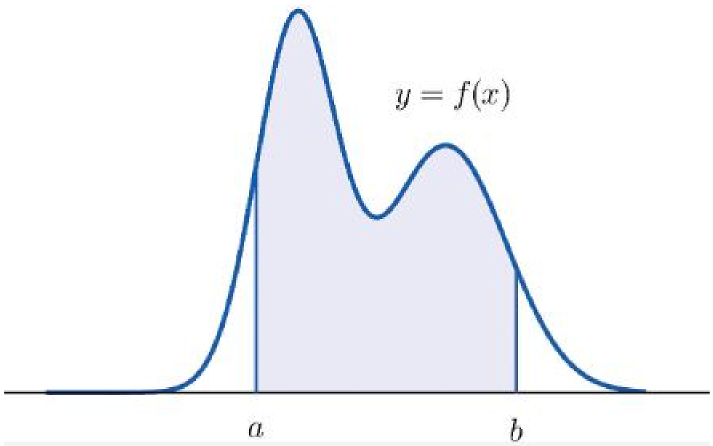
$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Numerous continuous probability distributions exist, such as Normal distribution, Uniform distribution and Exponential distribution.



The perpendiculars erected at any two points a and b give the probability that x is between the points a and b.

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$$



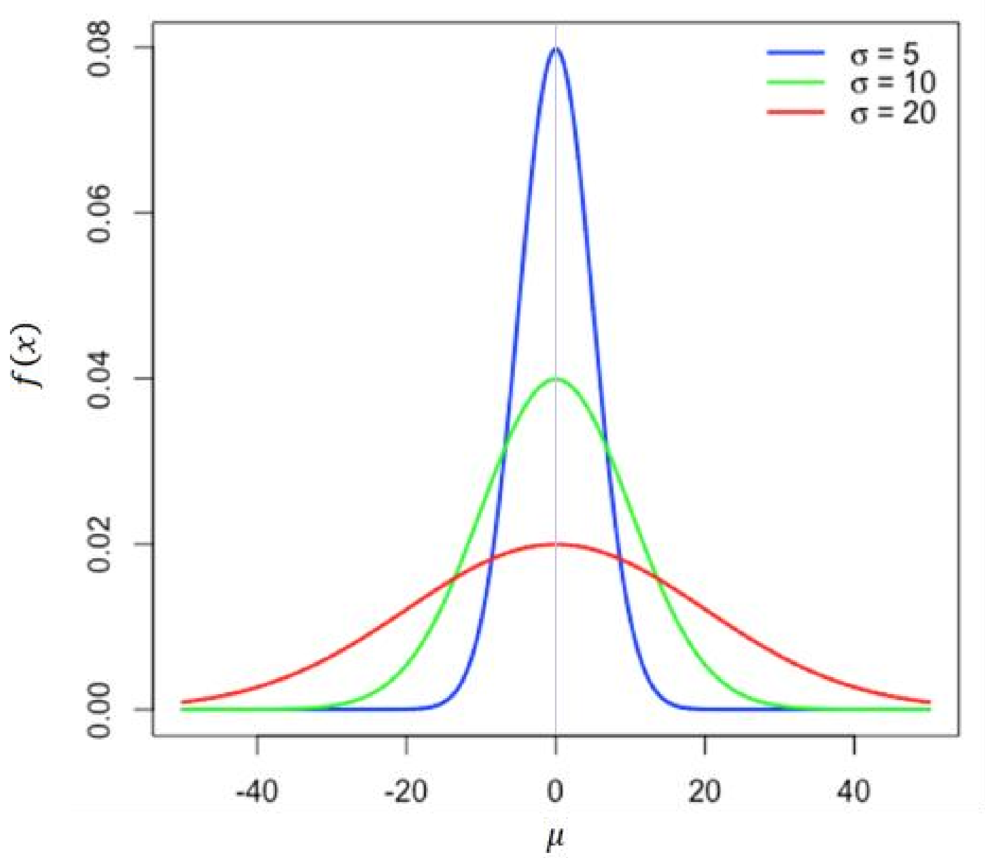
Graph of a continuous distribution showing area between a and b.

Normal Distribution

The Normal distribution, also known as the Gaussian distribution, is a fundamental distribution in statistics. For a continuous random variable x with mean μ and variance σ^2, denoted as $x \sim N\left(\mu, \sigma^2\right)$, the probability density function (PDF) $f(x)$ is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The Figure shows the normal distribution curves for $\mu = 0$ with standard deviations of 5, 10, and 20.



Curves of normal distribution

Properties of the Normal Distribution

1. **Symmetry:** The normal distribution is symmetric around its mean μ.
2. **Bell-shaped:** The curve has a single peak at the mean.
3. **Mean, Median, Mode:** All are equal (μ).
4. **Variance σ^2:** Controls spread; larger σ^2 means a wider curve.
5. **Total Area:** Under the curve = 1.
6. **Asymptotic:** Tails approach x-axis but never touch.

Standard Normal Distribution

The standard normal distribution, also called the Z-distribution, is a special normal distribution where the mean is 0 and the standard deviation is 1. That is $z \sim N(0,1)$.

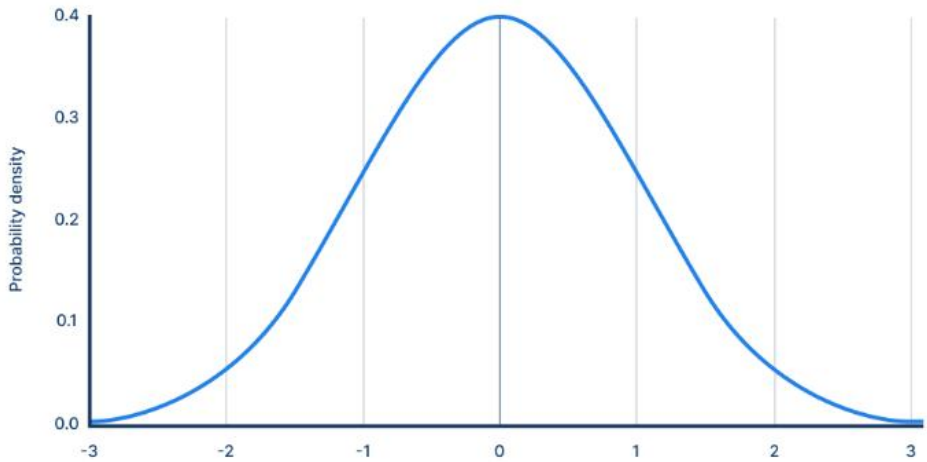
The PDF of the standard normal distribution $f(z)$ is given by:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

A normal distribution can be standardized by converting its values to z-scores using:

$$z = \frac{x - \mu}{\sigma}$$

The distribution plot below is a standard normal distribution.



- To calculate $P(x < x_1)$, when $x \sim N\left(\mu, \sigma^2\right)$.

1. Convert x_1 to a z-score:

$$z_1 = \frac{x_1 - \mu}{\sigma}$$

2. Use z-table to find $P(z < z_1)$, which equals $P(x < x_1)$.

Example 1: If $x \sim N(50,100)$, then find $P(x < 62)$.

Solution:

$$z_1 = \frac{62 - 50}{\sqrt{100}} = 1.2$$

$$P(x < 62) = P(z < 1.2)$$

From z-table

1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177

$P(z < 1.2) = 0.8849$ ⇨ $P(x < 62) = 0.8849$.

- To calculate $P(x_1 < x < x_2)$, when $x \sim N\left(\mu, \sigma^2\right)$.

1. $P(x_1 < x < x_2) = P(x < x_2) - P(x < x_1)$
2. Convert x_1 and x_2 to a z-score:

$$z_1 = \frac{x_1 - \mu}{\sigma} \text{ and } z_2 = \frac{x_2 - \mu}{\sigma}$$

3. Use z-table to find $P(z < z_1)$ and $P(z < z_2)$
4. $P(x_1 < x < x_2) = P(z < z_2) - P(z < z_1)$

Example 2: If $x \sim N(10,16)$, then find $P(8 < x < 12)$.

Solution: $P(8 < x < 12) = P(x < 12) - P(x < 8)$

$$z_1 = \frac{8 - 10}{4} = -1 \text{ and } z_2 = \frac{12 - 10}{4} = 1$$

From z-table $P(z < 1) = 0.8413$ and $P(z < -1) = 0.1587$

$$P(8 < x < 12) = P(z < 1) - P(z < -1) = 0.8413 - 0.1587 = 0.6826$$

- To calculate $P\left(x > x_1\right)$: $P\left(x > x_1\right) = 1 - P\left(x \leq x_1\right)$

Example 3: A medical physicist is analyzing the radiation doses delivered to patients undergoing radiotherapy. The doses follow a normal distribution with $\mu = 40$ Gy and $\sigma^2 = 9$ Gy. What is the probability that a patient receives a dose will be greater than 35 Gy?

Solution: $P(x > 35) = 1 - P(x \leq 35)$

$$z_1 = \frac{35 - 40}{3} = -1.67$$

$$P(x > 35) = 1 - P(z \leq -1.67) = 1 - 0.0475 = 0.9525$$

Example 4: If the hemoglobin level in the blood of a community is distributed according to the normal distribution with a mean of 16 and a variance of 0.81, then if we choose a person at random, what is the probability that his hemoglobin level will be greater than 17?

Solution:

$$z_1 = \frac{17 - 16}{0.9} = 1.11$$

$$P(x > 17) = 1 - P(z \leq 1.11) = 1 - 0.8665 = 0.1335$$

Example 5: Suppose the number of ridges in the fingerprints of individuals in a given population is normally distributed with mean 140 and standard deviation 50. Find the probability that the number of ridges on the fingers of an individual randomly selected from this population is: a) Less than 100. b) Between 120 and 200.

c) In a population of 10,000 people how many would you expect to have a ridge count of more than150?

Solution:

a) $z_1 = \frac{100-140}{50} = -0.8$

$$P(x < 100) = P(z < -0.8) = 0.2119$$

b) $P(120 < x < 200) = P(x < 200) - P(x < 120)$

$$z_1 = \frac{120-140}{50} = -0.4 \quad , \quad z_2 = \frac{200-140}{50} = 1.2$$

$$P(120 < x < 200) = P(z < 1.2) - P(z < -0.4) = 0.8849 - 0.3446 = 0.2023$$

c) $z_1 = \frac{150-140}{50} = 0.2$

$$P(x > 150) = 1 - P(z \leq 0.2) = 1 - 0.5793 = 0.4207$$

$$P(x > 150) = \frac{n(x > 150)}{n(S)}$$

$$0.4207 = \frac{n(x > 150)}{10000}$$

$$n(x > 150) = 4207$$

There are 4207 person have a ridge count of more than150.

H.W.

1. Let $x \sim N\ (75\ ,625)$. Then find:

(a) $P(50 \leq x \leq 100\ )$

(b) $P(x > 60\ )$

(c) $P(x < 90\ )$

2. If the total cholesterol values for a certain population are approximately normally distributed with a mean of 200 mg /100 ml and a standard deviation of 20 mg/100 ml, find the probability that an individual picked at random from this population will have a cholesterol values:

(a) Between 180 and 200 mg/100 ml

(b) Greater than 225 mg/100 ml

(c) Less than 150 mg/100 ml

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.5	0.0062	0.006	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.008	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.011
-2.1	0.0179	0.0174	0.017	0.0166	0.0162	0.0158	0.0154	0.015	0.0146	0.0143
-2	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.025	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.063	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.102	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.123	0.121	0.119	0.117
-1	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.166	0.1635	0.1611
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.305	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.281	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.33	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.352	0.3483
-0.2	0.4207	0.4168	0.4129	0.409	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
- 0	0.5	0.496	0.492	0.488	0.484	0.4801	0.4761	0.4721	0.4681	0.4641
0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952