

Partial Differential Equation PDE

A differential equation involving partial derivatives of a dependent variable (one or more) with more than one independent variable is called a partial differential equation, hereafter denoted as PDE. For example, the following equations are PDEs:

$$\begin{aligned} a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = f(x, t) &\equiv au_t + bu_x = f(x, t) \\ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 &\equiv u_{tt} - c^2 u_{xx} = 0 \quad (\text{Wave Equation}) \\ \frac{\partial u}{\partial t} - T \frac{\partial^2 u}{\partial x^2} = 0 &\equiv u_t - Tu_{xx} = 0 \quad (\text{Heat Equation}) \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 &\equiv u_{xx} + u_{yy} = 0 \quad (\text{Laplace Equation}) \end{aligned}$$

1. First Order PDE with Constant Coefficients

Consider the linear first order constant coefficient partial differential equation $au_t + bu_x = f(x, t)$ with an initial condition $u(x, 0) = g(x)$ where a and b are constants, $f(x, t)$ and $g(x)$ are given functions. This is a typical initial value problem for PDEs. Our objective is to find the solution $u(x, t)$.

Solution Formula $u(x, t) = g\left(x - \frac{b}{a}t\right) + \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}T, T\right) dT$

Example 1: Solve the initial value problem $u_t + 4u_x = 0$ with $u(x, 0) = 1/(1+x^2)$

Solution : $f(x, t) = 0$ and $g(x) = 1/(1+x^2)$ ($a = 1$, $b = 4$)

$$f(x, t) = 0 \Rightarrow \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}T, T\right) dT = 0$$

$$\text{So, } u(x, t) = g(x - 4t)$$

$$u(x, 0) = 1/(1+x^2) \Rightarrow g(x) = 1/(1+x^2)$$

$$\text{Then the solution of PDE is: } u(x, t) = \frac{1}{1+(x-4t)^2}$$

Example 2: Solve the initial value problem $2u_t + 3u_x = 0$ with $u(x, 0) = \frac{1+e^x}{1+e^{4x}}$

Solution: $f(x, t) = 0$ and $g(x) = \frac{1+e^x}{1+e^{4x}}$ ($a = 2$, $b = 3$)

$$f(x, t) = 0 \Leftrightarrow \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}T, T\right) dT = 0$$

$$\begin{aligned} g\left(x - \frac{b}{a}t\right) &= g\left(x - \frac{3}{2}t\right) = \frac{1+e^{x-3t/2}}{1+e^{4(x-3t/2)}} \\ &= \frac{1+e^{x-3t/2}}{1+e^{(4x-6t)}} \end{aligned}$$

$$u(x, t) = \frac{1+e^{x-3t/2}}{1+e^{(4x-6t)}}$$

Example 3: Solve the initial value problem $u_t + u_x = x$ with $u(x, 0) = \sin x$

Solution: $f(x, t) = x$ and $g(x) = \sin x$ ($a = 1$, $b = 1$)

$$\begin{aligned} f(x, t) = x \Leftrightarrow \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}T, T\right) dT &= \frac{1}{1} \int_0^t f\left(x - \frac{1}{1}t + \frac{1}{1}T, T\right) dT \\ &= \int_0^t (x - t + T) dT = \frac{(x - t + T)^2}{2} \Big|_0^t = \frac{x^2 - (x - t)^2}{2} \end{aligned}$$

$$g\left(x - \frac{b}{a}t\right) = g(x - t) = \sin(x - t)$$

$$u(x, t) = \sin(x - t) + \frac{x^2 - (x - t)^2}{2}$$

Example 4: Solve the initial value problem $u_t + 2u_x = 6te^{-2x}$ with $u(x, 0) = x^2$

Solution: $f(x, t) = 6te^{-2x}$ and $g(x) = x^2$ ($a = 1, b = 2$)

$$f(x, t) = 6te^{-2x} \Leftrightarrow \int_0^t f(x - 2t + 2T, T) dT = \int_0^t 6Te^{-2(x-2t+2T)} dT$$

+	$6T$	$e^{-2x+4t-4T}$
-	6	$\downarrow (-1/4)e^{-2x+4t-4T}$
+	0	$\downarrow (1/16)e^{-2x+4t-4T}$

$$\begin{aligned} \int_0^t 6Te^{-2(x-2t+2T)} dT &= \left[-\frac{3}{2}Te^{-2x+4t-4T} - \frac{3}{8}e^{-2x+4t-4T} \right]_0^t \\ &= -\frac{3}{2}te^{-2x} - \frac{3}{8}e^{-2x} + \frac{3}{8}e^{-2x+4t} \end{aligned}$$

$$g(x) = x^2 \Leftrightarrow g(x - 2t) = (x - 2t)^2$$

$$u(x, t) = (x - 2t)^2 - \frac{3}{2}te^{-2x} - \frac{3}{8}e^{-2x} + \frac{3}{8}e^{-2x+4t}$$

Example 5: Solve the initial value problem $2u_t + 5u_x = \frac{1}{4}\cos x + \frac{1}{6}\sin 4t$

with $u(x, 0) = e^{-x^2/8}$

Solution: $f(x, t) = \frac{1}{4}\cos x + \frac{1}{6}\sin 4t$ and $g(x) = e^{-x^2/8}$ ($a = 2, b = 5$)

$$\begin{aligned} u(x, t) &= g\left(x - \frac{b}{a}t\right) + f(x, t) = \frac{1}{4}\cos x + \frac{1}{6}\sin 4t \\ &\Leftrightarrow \frac{1}{a} \int_0^t f\left(x - \frac{b}{a}t + \frac{b}{a}T, T\right) dT = \frac{1}{2} \int_0^t f\left(x - \frac{5}{2}t + \frac{5}{2}T, T\right) dT \\ &= \frac{1}{2} \int_0^t \left(\frac{1}{4}\cos\left(x - \frac{5}{2}t + \frac{5}{2}T\right) + \frac{1}{6}\sin(4T) \right) dT \\ &= \frac{1}{2} \left[\frac{1}{10}\sin\left(x - \frac{5}{2}t + \frac{5}{2}T\right) - \frac{1}{24}\cos(4T) \right] \Big|_0^t \\ &= \frac{1}{20}\sin(x) - \frac{1}{48}\cos(4t) - \frac{1}{20}\sin\left(x - \frac{5}{2}t\right) + \frac{1}{48} \end{aligned}$$

$$g(x) = e^{-x^2/8} \Leftrightarrow g\left(x - \frac{b}{a}t\right) = g\left(x - \frac{5}{2}t\right) = e^{-(x-5t/2)^2/8} = e^{-(x-5t/2)^2/8}$$

$$u(x, t) = e^{-(x-5t/2)^2/8} + \frac{1}{20}\sin(x) - \frac{1}{48}\cos(4t) - \frac{1}{20}\sin\left(x - \frac{5}{2}t\right) + \frac{1}{48}$$

H.W: Solve the initial value problems

$$1. \quad 2u_t + 5u_x = 0 \quad \text{with } u(x, 0) = 1/(1 + e^x).$$

$$\text{Ans: } u(x, t) = 1/\left(1 + e^{x-\frac{5t}{2}}\right)$$

$$2. \quad 2u_t + 3u_x = \frac{1}{4}\sin x \quad \text{with } u(x, 0) = \frac{1 + e^x}{1 + e^{4x}}$$

$$\text{Ans: } u(x, t) = \frac{1 + e^{x-3t/2}}{1 + e^{4(x-3t/2)}} - \frac{1}{12}\cos x + \frac{1}{12}\cos\left(x - \frac{3t}{2}\right)$$

$$3. \quad 3u_t + 2u_x = \cos t \quad \text{with } u(x, 0) = \sin x$$

$$\text{Ans: } u(x, t) = \frac{1}{3}\sin t + \sin\left(x - \frac{2}{3}t\right)$$