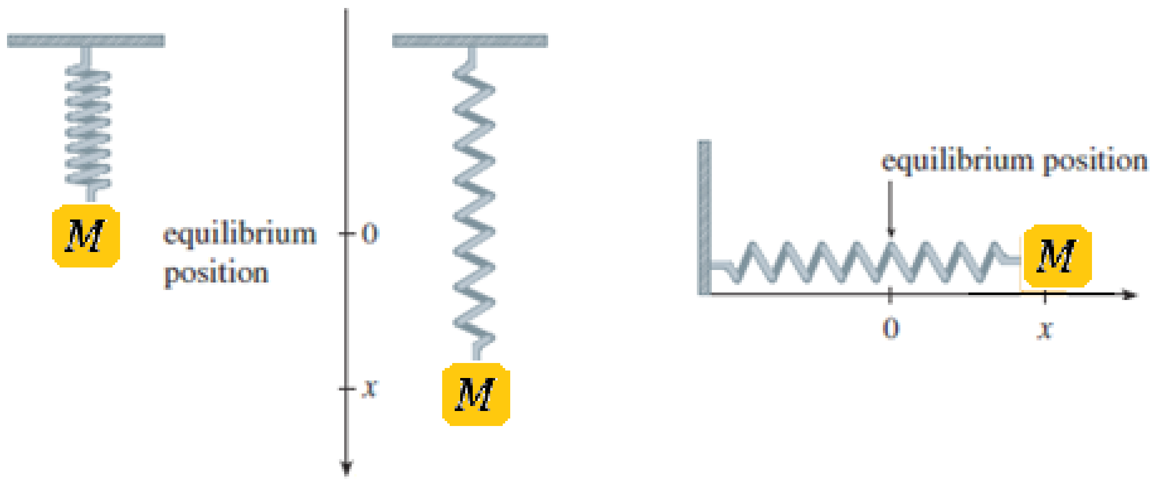
Physical Applications of 2nd ODE

1. Simple Harmonic Motion

Simple Harmonic Motion is defined as a motion in which the restoring force is directly proportional to the displacement of the body from its position. We consider the motion of an object with mass M at the end of a spring that is either vertical or horizontal on a level surface as in a Figure.



Hooke's Law, says that if the spring is stretched units from its natural length, then it exerts a force that is proportional to x.

Restoring Force $F = -kx$, where $k > 0$ is the spring constant.

If we ignore any external resisting forces then, by Newton's Second Law of motion we have $F = Ma$.

But $a = \dfrac{d^2x}{dt^2}$ ⇨ $F = M\dfrac{d^2x}{dt^2}$

$$M\frac{d^2x}{dt^2} = -kx$$

$$\boxed{M\frac{d^2x}{dt^2} + kx = 0 \text{ or } Mx'' + kx = 0}; \ x'' = \frac{d^2x}{dt^2}$$

Example 1: A frictionless spring with a $10kg$ mass can be held stretched $1\ m$ beyond its natural length by a force of $40\ N$. If the spring begins at its equilibrium position, but a push gives it an initial velocity of $2.5\ m/sec$, find the position of the mass after t seconds.

Solution: From Hooke's Law, the force required to stretch the spring is:

$$40 = 1k \quad \Rightarrow \quad k = 40\ N/m$$

$$10x'' + 40x = 0 \quad \Rightarrow \quad x'' + 4x = 0 \ \text{ with } \ x(0) = 1 \text{ and } x'(0) = 2.5$$

$$m^2 + 4 = 0 \Rightarrow m = \mp 2i$$

$$x = c_1 \sin 2t + c_2 \cos 2t$$

$$x(0) = 1 \quad \Rightarrow c_2 = 1$$

$$x' = 2c_1 \cos 2t - 2c_2 \sin 2t$$

$$x'(0) = 2.5 \Rightarrow 2.5 = 2c_1 \Rightarrow c_1 = 1.25$$

$$x = 1.25 \sin 2t + \cos 2t$$

Example 2: A spring with a mass of $2\ kg$ has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7m$. If the spring is stretched to a length of $0.7\ m$ and then released with initial velocity 0, find the position of the mass at any time.

Solution: Stretch on length $= 0.7 - 0.5 = 0.2\ m$

$$25.6 = 0.2k \Rightarrow k = 128\ N/m$$

$$2x'' + 128x = 0 \quad \Rightarrow \quad x'' + 64x = 0 \ \text{ with } \ x(0) = 0.2 \text{ and } x'(0) = 0$$

$$m^2 + 64 = 0 \Rightarrow m = \mp 8i$$

$$x = c_1 \sin 8t + c_2 \cos 8t$$

$$x(0) = 0.2 \quad \Rightarrow c_2 = 0.2$$

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t$$

$$x'(0) = 0 \Rightarrow 0 = 8c_1 \Rightarrow c_1 = 0$$

$$x = 0.2 \cos 8t$$

2. Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force. An example is the damping force supplied by a shock absorber in a car or a bicycle. We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Then

damping force $= -c\dfrac{dx}{dt}$, where $c > 0$ is the damping constant.

Thus, in this case, Newton's Second Law gives:

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

Example 3: A spring with a mass of $2\ kg$ has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. If the spring is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6\ m/s$.

Solution: $25.6 = 0.2k \Rightarrow k = 128$, $M = 2$ and $c = 40$

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

$$2x'' + 40x' + 128x = 0 \quad ; \quad x(0) = 0 \text{ and } x'(0) = 0.6$$

$$x'' + 20x' + 64x = 0$$

$$m^2 + 20m + 64 = 0 \Rightarrow (m+4)(m+16) = 0 \Rightarrow m = -4, -16$$

$$x(t) = c_1e^{-4t} + c_2e^{-16t}$$

$$x(0) = 0 \Rightarrow c_1 + c_2 = 0 \quad \cdots (1)$$

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

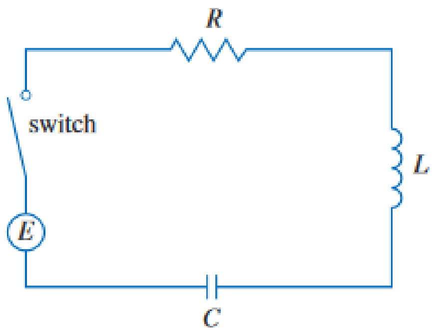
$$-4c_1 - 16c_2 = 0.6 \qquad \cdots (2)$$

$$c_1 = 0.05 \text{ and } c_2 = -0.05$$

$$x(t) = 0.05\ (e^{-4t} - e^{-16t})$$

3. Electric Circuits

The circuit shown in Figure contains an electromotive force E, a resistor R, an inductor L, and a capacitor C, in series. If the charge on the capacitor at time t is $Q(t)$, then the current is the rate of change of Q with respect to t : $I = dQ/dt$.



It is known from physics that the voltage drops across the resistor, inductor, and capacitor are: RI, $L\frac{dI}{dt}$ and $\frac{Q}{C}$ respectively.

Kirchhoff's voltage law says that the sum of these voltage drops is equal to the supplied voltage

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Since $I = dQ/dt$, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

Example 4: A series circuit consists of a resistor with $R = 20\ \Omega$, an inductor with $L = 1\ H$, a capacitor with $C = 0.002\ F$, and a $12 - V$ battery. If the initial charge and current are both 0, find the charge at time t.

. Solution:

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + \frac{Q}{0.002} = 12$$

$$\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 500\ Q = 12$$

$$m^2 + 20m + 500 = 0$$

$$m = \frac{-20 \mp \sqrt{400 - 2000}}{2} = -10 \mp 20i$$

$$Q_h = e^{-10t}(c_1 \sin 20t + c_2 \cos 20t)$$

Now, $Q_p = A \Rightarrow \dfrac{dQ}{dt} = \dfrac{d^2Q}{dt^2} = 0$

So $500\, Q_p = 12 \quad \Rightarrow \quad Q_p = \dfrac{3}{125}$

$$Q(t) = e^{-10t}(c_1 \sin 20t + c_2 \cos 20t) + \frac{3}{125}$$

$Q(0) = 0 \quad \Rightarrow \quad 0 = c_2 + \dfrac{3}{125} \quad \Rightarrow \quad c_2 = -\dfrac{3}{125}$

$$I = \frac{dQ}{dt} = e^{-10t}(20c_1 \cos 20t - 20c_2 \sin 20t) - 10e^{-10t}(c_1 \sin 20t + c_2 \cos 20t)$$

$I(0) = 0 \Rightarrow \quad 0 = 20c_1 - 10c_2 \quad \Rightarrow c_1 = -\dfrac{3}{250}$

$$Q(t) = e^{-10t}\left(-\frac{3}{250}\sin 20t - \frac{3}{125}\cos 20t\right) + \frac{3}{125}$$

Exercises

1. A spring with a 3 kg mass is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds.

2. A spring with a mass of 3 kg has damping constant $c = 30$ and spring constant $k = 123$. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of 2 m/s.

3. A series circuit consists of a resistor with $R = 24\ \Omega$, an inductor with $L = 2\ H$, a capacitor with $C = 0.005\ F$, and a $12 - V$ battery. If $Q(0) = 0.001$ and $I(0) = 0$, find the charge at time t.