Example (1):

Evaluate the commutator $\left[x, \frac{d}{dx}\right]$, applying the commutator on an arbitrary function *f(x)*

Solution:

$\left[x, \frac{d}{dx}\right] F(x) = x \frac{d}{dx} f(x) - \frac{d}{dx}(x f(x))$

$= x \frac{df(x)}{dx} - x \frac{df(x)}{dx} - f(x) = - f(x)$

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$\left[x, \frac{d}{dx}\right] F(x) = - f(x)$

$\left[x, \frac{d}{dx}\right] = -1$

From the Angular momentum L of a particle having linear momentum p is:

$L = r \times P$

$$\therefore\ L = r \times P = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$L = i\left(y\, p_z - z\, p_y\right) + j(z\, p_x - x\, p_z) + k(x\, p_y - y\, p_x)$$

$\therefore L_x = y\ p_z - z\ p_y = -i\hbar\,(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y})$

$\therefore L_y = z\ p_x - x\, p_z = -i\hbar\,(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z})$

$\therefore L_z = x\, p_y - y\, p_x = -i\hbar\,(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x})$

Example (2):

Prove that the commutator $[x, p_x] = i\hbar$

Solution:

$$[x, p_x]\,\Psi_{(x)} = (x\,p_x - p_x\,x)\Psi_{(x)} = (-i\hbar\, x\,\frac{\partial}{\partial x} + i\hbar\,\frac{\partial}{\partial x}\,x\,)\,\Psi_{(x)}$$

$$[x, p_x]\,\Psi_{(x)} = i\hbar\,\{-x\,\frac{\partial \Psi_{(x)}}{\partial x} + \frac{\partial}{\partial x}\,(x\,\Psi_{(x)})\}$$

$$[x, p_x]\,\Psi_{(x)} = i\hbar\,\{-x\,\frac{\partial \Psi_{(x)}}{\partial x} + x\,\frac{\partial \Psi_{(x)}}{\partial x} + \Psi_{(x)}\frac{\partial x}{\partial x}\}$$

$$[x, p_x]\,\Psi_{(x)} = i\hbar\,\Psi_{(x)}$$

$$\therefore\ [x, p_x] = i\hbar$$

Similarly we can prove that

$$[y, p_y] = [z, p_z] = i\hbar$$

$$\therefore\ [x, p_x] = [y, p_y] = [z, p_z] = i\hbar$$

And

$$[p_x, x] = [p_y, y] = [p_z, z] = -\,i\hbar$$

Example (3):

Prove that $[x, p_y] = 0$

Solution:

$$[x, p_y]\,\Psi_{(x)} = (x\,p_y - y\,p_x\,)\Psi_{(x)} = (-i\hbar\, x\,\frac{\partial}{\partial y} + i\hbar\,\frac{\partial}{\partial y}\,x\,)\,\Psi_{(x)}$$

$$[x, p_y]\,\Psi_{(x)} = i\hbar\,\{-x\,\frac{\partial \Psi_{(x)}}{\partial y} + \frac{\partial}{\partial y}\,(x\,\Psi_{(x)})\}$$

$\left[x, p_y\right] \Psi_{(x)} = i\hbar \{ -x \frac{\partial \Psi_{(x)}}{\partial y} + x \frac{\partial \Psi_{(x)}}{\partial y} + \Psi_{(x)} \frac{\partial x}{\partial y} \} = 0$

$\left[x, p_y\right] \Psi_{(x)} = 0$

$\therefore \left[x, p_y\right] = 0$

Similarly we can prove that:

$$[x, p_z] = [y, p_x] = [y, p_z] = [z, p_x] = \left[z, p_y\right] = 0$$

Example (4):

For angular momentum operator L, show that $[L_x, L_y] = i\hbar L_z$

Solution:

$[L_x, L_y] = \left[\left(y\, p_z - z\, p_y\right), (zp_x - xp_z)\right]$

$[L_x, L_y] = [y\, p_z, zp_x] - [y\, p_z, xp_z] - \left[z\, p_y, zp_x\right] + \left[z\, p_y, xp_z\right]$

$[L_x, L_y] = y\, p_x[p_z, z] - y\, p_z[p_z, x] - z\, p_x\left[p_y, z\right] + x\, p_y[z, p_z]$

$[L_x, L_y] = y\, p_x[p_z, z] + x\, p_y[z, p_z]$

But $[p_z, z] = -i\hbar$, and $[z, p_z] = i\hbar$

Therefor

$[L_x, L_y] = y\, p_x(-i\hbar) + x\, p_y\, (i\hbar) = i\hbar x p_y - i\hbar y p_x$

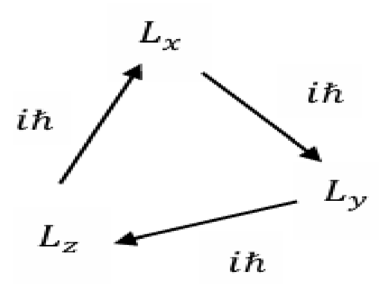
$[L_x, L_y] = i\hbar\, (xp_y - yp_x)$

$[L_x, L_y] = i\hbar L_z$

Similarly we can prove that

$[\,L_y, L_z] = i\hbar L_x$

$[\,L_z, L_x] = i\hbar L_y$



Example (5):

Evaluate the commutator:

$[x^2,\frac{d}{dx}]$ applying the commutator on an arbitrary function *f(x)*.

Solution:

$[x^2,\frac{d}{dx}]\, f_{(x)} = x^2 \frac{d}{dx} f_{(x)} - \frac{d}{dx}\,(x^2 f_{(x)})$

$\frac{d}{dx}\,(x^2 f_{(x)}) = 2\,x\,f_{(x)} + x^2 \frac{df_{(x)}}{dx}$

So:

$x^2 \frac{d}{dx} f_{(x)} - (2\,x\,f_{(x)} + x^2 \frac{df_{(x)}}{dx})$

$= -2x\, f_{(x)}$

$\therefore \left[x^2,\frac{d}{dx}\right] = -2x$