Partial Derivatives

Partial derivatives are a fundamental concept in multivariable calculus. They measure how a function changes when one of its input variables changes, while keeping the other variables constant. For a function with two independent variables, $z = f(x, y)$, the partial derivative with respect to one variable is found by differentiating normally, treating the other variable as a constant. The same applies to the partial derivative with respect to the other variable. For functions with multiple variables, this process is called partial differentiation, where you differentiate with respect to one variable while keeping the others constant.

We usually denote the partial derivative using one of the following symbols:

The partial derivative of f with respect to x represented by f_x or $\frac{\partial f}{\partial x}$

The partial derivative of f with respect to y represented by f_y or $\frac{\partial f}{\partial x}$

Example 1: Find f_x and f_y for the functions:

1. $f(x, y) = x^2y^4$ ⇨ $f_x = 2xy^4$ and $f_y = 4x^2y^3$
2. $f(x, y) = x^3 + y^2$ ⇨ $f_x = 3x^2$ and $f_y = 2y$
3. $f(x, y) = e^{2y+3}\sin 3x$ ⇨ $f_x = 3e^{2y+3}\cos 3x$ and $f_y = 2e^{2y+3}\sin 3x$

Example 2: Find w_x and w_y for $w(x, y) = x^2 \sin(xy)$

$$w_x = x^2(\cos(xy)) \times y + 2x \sin(xy) = x^2y \cos(xy) + 2x \sin(xy)$$

$$w_y = x^2(\cos(xy)) \times x = x^3 \cos(xy)$$

Example 3: Find h_s for $h(s, t) = t\ln(4s^2 + 1) + t^2 \tan^{-1}(2s)$

$$h_s = \frac{8st}{4s^2 + 1} + \frac{2t^2}{1 + 4s^2} = \frac{8st + 2t^2}{4s^2 + 1}$$

Example 4: If $w = x\sin(yz) + xe^{yz}$, then show that 1. $xw_x = w$ 2. $yw_y = zw_z$

1. $w_x = \sin(yz) + e^{yz}$ ⇨ $xw_x = x\sin(yz) + xe^{yz} = w$
2. $w_y = xz\cos(yz) + xze^{yz}$ ⇨ $yw_y = xyz\cos(yz) + xyze^{yz}$

$w_z = xy\cos(yz) + xye^{yz}$ ⇨ $zw_z = xyz\cos(yz) + xyze^{yz}$

Second Order Partial Derivatives

Let $z = f(x, y)$ be a function of x and y, then the second partial derivative of f with respect to x is f_{xx} , the second partial derivative of f with respect to y is f_{yy} , the second partial derivative of f with respect to y and then with respect to x is f_{xy} and the second partial derivative of f with respect to x and then with respect to y is f_{yx}.

$$\left(f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}, f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} \text{ and } f_{yx} \equiv \frac{\partial^2 f}{\partial y \partial x}\right)$$

f_{xy} and f_{yx} are called mixed partial derivatives where $f_{xy} = f_{yx}$.

Example 5: Find f_{xx} , f_{xy} , f_{yx} and f_{yy} for $f(x, y) = x^2 + 3xy - y^4$

$$f_x = 2x + 3y \quad \Rightarrow \quad f_{xx} = 2 \quad \text{and} \quad f_{yx} = 3$$

$$f_y = 3x - 4y^3 \quad \Rightarrow \quad f_{yy} = -12y^2 \quad \text{and} \quad f_{xy} = 3$$

Example 6: Find f_{rr} and $f_{\theta\theta}$ for $f(r, \theta) = r^2 \sin^2 \theta$

$$f_r = 2r \sin^2 \theta \quad \Rightarrow \quad f_{rr} = 2 \sin^2 \theta$$

$$f_\theta = 2r^2 \sin \theta \cos \theta \quad \text{but} \quad (2 \sin \theta \cos \theta = \sin 2\theta)$$

$$f_\theta = r^2 \sin 2\theta \quad \Rightarrow \quad f_{\theta\theta} = 2r^2 \cos 2\theta$$

Example 7: Find all first and second order partial derivatives of the function

$$f(x, y, z) = 3x^2 - 2xy^2 + 4x^2 z + z^3 y + 5$$

$$f_x = 6x - 2y^2 + 8xz \quad \Rightarrow \quad f_{xx} = 6 + 8z, \quad f_{yx} = -4y \text{ and } f_{zx} = 8x$$

$$f_y = -4xy + z^3 \quad \Rightarrow \quad f_{yy} = -4x, \quad f_{xy} = -4y \text{ and } f_{zy} = 2z^2$$

$$f_z = 4x^2 + 2z^2 y \quad \Rightarrow \quad f_{zz} = 4zy, \quad f_{xz} = 8x \quad \text{and } f_{yz} = 2z^2$$

Chain Rule for Partial Derivatives

If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, then $z = z(u, v)$ and

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}}$$

Example 8: If $z = x^2 + y^2, x = 2u + v, y = 2v - u$.

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a functions of u and v

$$\frac{\partial z}{\partial x} = 2x = 4u + 2v, \qquad \frac{\partial z}{\partial y} = 2y = 4v - 2u$$

$$\frac{\partial x}{\partial u}=2\ ,\quad \frac{\partial y}{\partial u}=-1,\quad \frac{\partial x}{\partial v}=1 \quad \text{and} \quad \frac{\partial y}{\partial v}=2$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial u}$$

$$=(4u+2v)\times 2+(4v-2u)\times(-1)$$

$$=8u+4v-4v+2u=10u$$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\times\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\times\frac{\partial y}{\partial v}$$

$$=(4u+2v)\times 1+(4v-2u)\times 2$$

$$=4u+2v+8v-4u=10v$$

Example 9: Let $w=xy+z,\ x=\sin t,\ y=\cos t$ and $z=t$. Find $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial t}=\frac{\partial w}{\partial x}\times\frac{\partial x}{\partial t}+\frac{\partial w}{\partial y}\times\frac{\partial y}{\partial t}+\frac{\partial w}{\partial z}\times\frac{\partial z}{\partial t}$$

$$=y\times\cos t+x\times(-\sin t)+(1)\times 1$$

$$=\cos^2 t-\sin^2 t+1=2\cos^2 t$$

Laplace's Equation: We say that the function $f(x,y)$ satisfies Laplace's equation if

$$\boxed{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0} \qquad (f_{xx}+f_{yy}=0)$$

Example 10: Show that the functions satisfy Laplace's equation

1. $f(x,y)=e^{-2y}\cos 2x$ 2. $w(s,t)=\ln(t^2+s^2)$

1. $\frac{\partial f}{\partial x}=-2e^{-2y}\sin 2x \quad ⇨ \quad \frac{\partial^2 f}{\partial x^2}=-4e^{-2y}\cos 2x$

$\frac{\partial f}{\partial y}=-2e^{-2y}\cos 2x \quad ⇨ \quad \frac{\partial^2 f}{\partial y^2}=4e^{-2y}\cos 2x$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=-4e^{-2y}\cos 2x+4e^{-2y}\cos 2x=0$$

2. $\frac{\partial w}{\partial s}=\frac{2s}{t^2+s^2} \quad ⇨ \quad \frac{\partial^2 w}{\partial s^2}=\frac{2(t^2+s^2)-2s\times 2s}{(t^2+s^2)^2}$

$$\frac{\partial^2 w}{\partial s^2}=\frac{2t^2+2s^2-4s^2}{(t^2+s^2)^2}=\frac{2t^2-2s^2}{(t^2+s^2)^2}$$

$$\frac{\partial w}{\partial t}=\frac{2t}{t^2+s^2} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial t^2}=\frac{2(t^2+s^2)-2t\times 2t}{(t^2+s^2)^2}$$

$$\frac{\partial^2 w}{\partial t^2}=\frac{2t^2+2s^2-4t^2}{(t^2+s^2)^2}=\frac{2s^2-2t^2}{(t^2+s^2)^2}$$

$$\frac{\partial^2 w}{\partial s^2}+\frac{\partial^2 w}{\partial t^2}=\frac{2t^2-2s^2}{(t^2+s^2)^2}+\frac{2s^2-2t^2}{(t^2+s^2)^2}=0$$

The 1-D Heat Equation: The 1-D Heat equation takes the form:

$$\boxed{\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2}}, \quad \text{where } k>0 \text{ which is called the thermal diffusivity.}$$

Example 11: Show that the function $T(x,t)=3e^{-4\pi^2 t}\cos(2\pi x)$ satisfy heat equation, with $k=1$.

$$\frac{\partial T}{\partial t}=-12\pi^2 e^{-4\pi^2 t}\cos(2\pi x)$$

$$\frac{\partial T}{\partial x}=-6\pi e^{-4\pi^2 t}\sin(2\pi x)$$

$$\frac{\partial^2 T}{\partial x^2}=-12\pi^2 e^{-4\pi^2 t}\cos(2\pi x)$$

$$\therefore \quad \frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}$$

Example 12: If $T(x,t)=2e^{-12t}\sin 2x$ satisfy heat equation, then find the thermal diffusivity k.

$$\frac{\partial T}{\partial t}=-24e^{-12t}\sin 2x$$

$$\frac{\partial T}{\partial x}=4e^{-12t}\cos 2x \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2}=-8\,e^{-12t}\sin 2x$$

$$\frac{\partial T}{\partial t}=k\frac{\partial^2 T}{\partial x^2} \quad \Rightarrow \quad -24e^{-12t}\sin 2x=-8ke^{-12t}\sin 2x$$

$$\therefore \quad k=3$$

Wave Equation: The wave equation takes the form: $\boxed{u_{tt} = c^2 u_{xx}}$

Example 13: Show that the function $u(x,t) = \cos(x + 2t) - \cos(x - 2t)$ satisfy the wave equation $u_{tt} = 4u_{xx}$

$$u_t = -2\sin(x + 2t) - 2\sin(x - 2t)$$

$$u_{tt} = -4\cos(x + 2t) + 4\cos(x - 2t)$$

$$u_x = -\sin(x + 2t) - \sin(x - 2t)$$

$$u_{xx} = -\cos(x + 2t) + \cos(x - 2t)$$

$$\therefore \; u_{tt} = 4u_{xx}$$

H.W.

1. If $w = \cos(x + y) + \sin(x - y)$, then show that $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial y^2}$.

2. Find $\dfrac{\partial^2 z}{\partial x \partial y}$ if $z = x^2 \sin(2x - 3y)$.

3. If $z = \ln(xy)$ and $x = r\sin\theta$, $y = r\cos\theta$, then find $\dfrac{\partial z}{\partial \theta}$

4. If $w = x^2 y + xy^2 + 2x^3 z^2$, then find $\dfrac{\partial w}{\partial x}$ at $x = 1, y = 2$ and $z = -1$.

5. Show that the functions satisfy Laplace's equation

   $a)\; f(x,y) = e^{3x}\sin 3y$ $\qquad$ $b)\; f(x,y) = x^3 - 3xy^2$

6. If $T(x,t) = 5e^{-32\pi^2 t}\sin(4\pi x)$ satisfy heat equation, then find the thermal diffusivity k.

7. Show that the function $u(x,t) = 3\sin(2x + 4t)$ satisfy wave equation $u_{tt} = 4u_{xx}$.