2. Linear ODE

A first-order linear differential equation is an equation of the form $y' + P(x)y = Q(x)$, where P and Q are functions of x. This form is called the **standard form** of a linear ODE. To solve a linear ODE, keep to the following steps:

1. Write ODE in **standard form** to identify the functions $P(x)$ and $Q(x)$.
2. Find **integrating factor** $\mu(x)$, where $\boxed{\mu(x) = e^{\int P(x)dx}.}$
3. Find the **general solution** of a linear ODE which is given by:

$$\boxed{y\,\mu(x) = \int \mu(x)\ Q(x)dx.}$$

Example 1: Find the general solution of $y' + 3y = 5e^{2x}$

Solution: The ODE is linear, in the **standard form**, with $P(x) = 3$ and $Q(x) = 5e^{2x}$.

So, the **integrating factor** is: $\mu(x) = e^{\int P(x)dx} = e^{\int 3dx} = e^{3x}$

Then the **general solution**: $y\,e^{3x} = \int 5e^{3x}\ e^{2x}dx = \int 5e^{5x}\ dx$

$$ye^{3x} = e^{5x} + C \quad \Rightarrow \quad y = e^{2x} + Ce^{-3x}$$

Example 2: Solve the Initial Value Problem (IVP) $\dfrac{dv}{dt} + 0.196v = 9.8$ with $v(0) = 48$.

Solution: The ODE is linear, in the **standard form**, with $P(t) = 0.196$ and $Q(t) = 9.8$

Integrating factor: $\mu(t) = e^{\int P(t)dt} = e^{\int 0.196dt} = e^{0.196t}$

The general solution: $ve^{0.196t} = \displaystyle\int 9.8\, e^{0.196t}\,dt = \frac{9.8}{0.196}\, e^{0.196t} + C$

$$ve^{0.196t} = 50\, e^{0.196t} + C \quad \Rightarrow \quad v = 50 + Ce^{-0.196t}$$

To find C, we have $v(0) = 48$

$48 = 50 + C \ \Rightarrow\ C = -2$

$v = 50 - 2e^{-0.196t}$

Example 3: Find the general solution of $xy' - 2y = x^2$. Assume $x > 0$.

Solution: Begin by writing the ODE in standard form

$$y' + \left(\frac{-2}{x}\right)y = x. \qquad \text{Standard form,} \quad y' + P(x)y = Q(x)$$

$$P(x) = \frac{-2}{x} \quad \text{and} \quad Q(x) = x.$$

Integrating factor: $\mu(x) = e^{\int P(x)dx} = e^{\int -2/x dx}$

$$\mu(x) = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2}$$

This implies that the general solution is

$$yx^{-2} = \int x^{-2}\ x\, dx$$

$$yx^{-2} = \int \frac{1}{x}\ dx = \ln x + C$$

$$y = x^2 \ln x + Cx^2 \qquad \text{(General solution)}$$

Example 4: Find the solution of IVP $x^2y' + xy = 1;\quad x > 0,\quad y(1) = 5$.

Solution: Standard form: $y' + \frac{1}{x}y = \frac{1}{x^2}$

$$P(x) = \frac{1}{x} \quad \text{and} \quad Q(x) = \frac{1}{x^2}.$$

Integrating factor: $\mu(x) = e^{\int P(x)dx} = e^{\int 1/x dx} = e^{\ln x} = x$

General solution: $yx = \int x\,\frac{1}{x^2}dx = \int \frac{1}{x}dx$

$$yx = \ln x + C$$

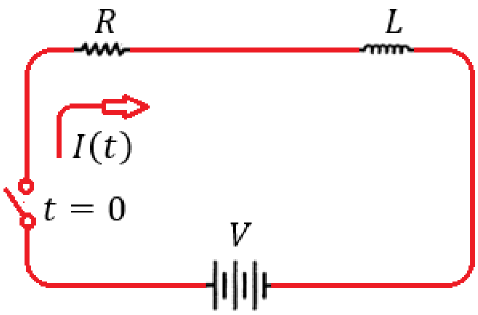
$$y = \frac{1}{x}(\ln x + C)$$

$$y(1) = 5 \ \Rightarrow\ 5 = 1(0 + C) \ \Rightarrow\ C = 5$$

$$y = \frac{1}{x}(\ln x + 5)$$

Electric Circuits

The RL circuit shown in the figure below has a **resistor** $R\ ohms$ and an **inductor** $L\ henries$ connected in series. A constant **voltage** $V\ volts$ is applied when the switch is closed.



RL circuit diagram

The voltage across the **resistor** is given by:

$$V_R = RI.$$

The voltage across the **inductor** is given by:

$$V_L = L\frac{dI}{dt}$$

Kirchhoff's voltage law states that the algebraic sum of all voltages around any closed loop in a circuit is zero. This results in the following differential equation:

$$L\frac{dI}{dt} + RI = V.$$

Example 5: In RL electric circuit $V = 12\ volts,\ \ L = 0.25\ henries$ and $R = 15\ ohms.$ Find current $I = I(t)$ at any time t.

Solution: $\left\{0.25\frac{dI}{dt} + 15\,I = 12\right\} \times 4$

Standard form: $\frac{dI}{dt} + 60\,I = 48,$ which is a linear ODE

$P(t) = 60$ and $Q(t) = 48.$

Integrating factor: $\mu(t) = e^{\int P(t)dt} = e^{\int 60dt} = e^{60t}$

General solution: $I\,e^{60t} = \int 48\,e^{60t}\,dt = \frac{48}{60}e^{60t} + C$

$$I(t) = \frac{4}{5} + Ce^{-60t}$$

When $t = 0$ the value of $I = 0$

$$0 = \frac{4}{5} + Ce^{-60\times 0} \quad ⇨ \quad C = -\frac{4}{5}$$

$$I(t) = \frac{4}{5}(1 - e^{-60t})$$

H.W

(1) Solve the ODEs:

(a) $x\,y' - 3y = x^2.$ (b) $x\,y' + y = \sin x.$

(2) Solve the IVPs:

(a) $y' + y\tan x = \sec x$ with $y(\pi/4) = \sqrt{2}.$

(b) $\frac{dv}{dt} + 2v = e^{-t}$ with $v(0) = 5.$

(3) In RL electric circuit, if $E = 10v$, $L = 0.5H$ and $R = 12\Omega$. Find current $I(t)$ at any time t.

Websites:

1. https://www.sfu.ca/math-coursenotes/Math%20158%20Course%20Notes/sec_first_order_homogeneous_linear.html
2. https://www.intmath.com/differential-equations/5-rl-circuits.php