# Applications of Partial Derivatives

## 1. Jacobian and Hessian Matrices:

The **Jacobian Matrix** is a matrix composed of the first-order partial derivatives of a multivariable function $f(x,y) = \big(u(x,y),\ v(x,y)\big)$. The formula for the Jacobian matrix is the following:

$$\mathrm{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

The determinant of Jacobian matrix is denoted by $|\mathrm{J}| = u_x v_y - v_x u_y$

**Example 1:** Let $u = x^2 - y^2, v = 2xy$. Find Jacobian matrix and its determinant.

$$u_x = 2x, u_y = -2y, v_x = 2y \text{ and } v_y = 2x$$

$$\mathrm{J} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$|\mathrm{J}| = 2x \times 2x - 2y \times (-2y) = 4x^2 + 4y^2$$

The Jacobian matrix of the function with 3 variables $f(x,y,z) = (u,v,w)$

$$\mathrm{J} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

**Example 2:** Let $u = x^2y, v = y^2z,\ \ w = xz^2$. Find Jacobian matrix and its determinant at the point (1,1,1).

$$\mathrm{J} = \begin{bmatrix} 2xy & x^2 & 0 \\ 0 & 2yz & y^2 \\ z^2 & 0 & 2xz \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$|\mathrm{J}| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} \begin{matrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{matrix}$$

$$= 2 \times 2 \times 2 + 1 \times 1 \times 1 + 0 \times 0 \times 0 - (1 \times 2 \times 0 + 0 \times 1 \times 2 + 2 \times 0 \times 1)$$

$$= 9$$

The second-order partial derivatives of the function $f(x, y)$ can be arranged as a matrix called the **Hessian Matrix**, denoted by $H(f)$:

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The determinant of Hessian matrix is denoted by $|H(f)| = f_{xx}f_{yy} - f_{xy}f_{yx}$

**Example 3:** Find Hessian matrix and its determinant at the point $(1,0)$.

for the function $f(x, y) = (x^2 + y^2)^2/2$

$$f_x = 2x(x^2 + y^2) \Rightarrow \quad f_{xx} = 2x \times 2x + 2(x^2 + y^2) = 6x^2 + 2y^2 \quad \Rightarrow f_{xx}|_{(1,0)} = 6$$

$$f_y = 2y(x^2 + y^2) \Rightarrow \quad f_{yy} = 2y \times 2y + 2(x^2 + y^2) = 2x^2 + 6y^2 \quad \Rightarrow f_{yy}|_{(1,0)} = 2$$

$$f_{xy} = f_{yx} = 2y \times 2x = 4xy \quad \Rightarrow \quad f_{xy}|_{(1,0)} = 0$$

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \quad \Rightarrow \quad |H(f)| = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12$$

The Hessian matrix of the function with 3 variables $f(x, y, z)$

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

**Example 4:** Let $f(x, y, z) = x^2e^y + xz^2$. Find Hessian matrix and its determinant at the point $(1,0,2)$.

$$f_x = 2xe^y + z^2 \quad \Rightarrow \quad f_{xx} = 2e^y \quad \Rightarrow \quad f_{xx}|_{(1,0,1)} = 2$$

$$f_x = 2xe^y + z^2 \quad \Rightarrow \quad f_{xy} = 2xe^y \quad \Rightarrow \quad f_{xy}|_{(1,0,1)} = 2$$

$$f_x = 2xe^y + z^2 \quad \Rightarrow \quad f_{xz} = 2z \quad \Rightarrow \quad f_{xz}|_{(1,0,1)} = 4$$

$$f_y = x^2e^y \quad \Rightarrow \quad f_{yx} = 2xe^y \quad \Rightarrow \quad f_{yy}|_{(1,0,1)} = 2$$

$$f_y = x^2e^y \quad \Rightarrow \quad f_{yy} = x^2e^y \quad \Rightarrow \quad f_{yy}|_{(1,0,1)} = 1$$

$$f_y = x^2e^y \quad \Rightarrow \quad f_{yz} = 0 \quad \Rightarrow \quad f_{yz}|_{(1,0,1)} = 0$$

$$f_z = 2xz \quad \Rightarrow \quad f_{zx} = 2z \quad \Rightarrow \quad f_{zx}|_{(1,0,1)} = 2$$

$f_z = 2xz \quad \Rightarrow \quad f_{zy} = 0 \quad \Rightarrow \quad f_{zy}\big|_{(1,0,1)} = 0$

$f_z = 2xz \quad \Rightarrow \quad f_{zz} = 2x \quad \Rightarrow \quad f_{zz}\big|_{(1,0,1)} = 2$

$$H(f) = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

$$|H(f)| = \begin{vmatrix} 2 & 2 & 4 \\ 2 & 1 & 0 \\ 2 & 0 & 2 \end{vmatrix} \begin{matrix} 2 & 2 \\ 2 & 1 \\ 2 & 0 \end{matrix}$$

$$= 2 \times 1 \times 2 + 2 \times 0 \times 2 + 4 \times 2 \times 0 - (2 \times 1 \times 4 + 0 \times 0 \times 2 + 2 \times 2 \times 2)$$

$$= 4 - (8 - 8) = -12$$

## 2. Gradient and Laplace Operator of a Scalar Field

A scalar field is a function that takes a point in space and assign a number to it, for example $f(x, y, z) = x^2 + \cos 2y + \ln(2z + 1)$

$f(1, \pi/6, 0) = 1 + \cos(\pi/3) + \ln 1 = 1 + (1/2) + 0 = 3/2$

**The Gradient** of a scalar field $f(x, y, z)$ is a vector field denoted by $\nabla f$ and it is defined as: $\boxed{\nabla f = f_x\, i + f_y\, j + f_z\, k}$

**Example 5:** Find $\nabla f$ of $f(x, y, z) = 2x^2 \sin y - xy \tan z$

$$f_x = 4x \sin y - y \tan z, \quad f_y = 2x^2 \cos y - x \tan z \text{ and } f_z = -xy \sec^2 z$$

$$\nabla f = (4x \sin y - y \tan z)\, i + (2x^2 \cos y - x \tan z)\, j - xy \sec^2 z\ k$$

**Laplace Operator**: The differential operator $\nabla^2$ is called Laplace operator and it is defined as: $\boxed{\nabla^2 f = f_{xx} + f_{yy} + f_{zz}}$

**Example 6:** Find $\nabla f$ and $\nabla^2 f$ for $f(x, y, z) = x^3 e^y + xy^2 z^3$

$$f_x = 3x^2 e^y + y^2 z^3 \quad \Rightarrow \quad f_{xx} = 6x e^y$$

$$f_y = x^3 e^y + 2xyz^3 \quad \Rightarrow \quad f_{yy} = x^3 e^y + 2xz^3$$

$$f_z = 3xy^2 z^2 \quad \Rightarrow \quad f_{zz} = 6xy^2 z$$

$$\nabla f = f_x\, i + f_y\, j + f_z\, k = (3x^2 e^y + y^2 z^3)\, i + (x^3 e^y + 2xyz^3)\, j + 3xy^2 z^2\, k$$

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = 6x e^y + x^3 e^y + 2xz^3 + 6xy^2 z$$

## 3. Divergence and the Curl of a Vector Field

**The Divergence** of a vector field $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ is computed as:

$$div\ F = \nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

**Example 7:** Find $div\ F$ if $F(x,y,z) = xzi + e^{yz}j - \ln(xy)\,k$

$$div\ F = \nabla . F = \frac{\partial(xz)}{\partial x} + \frac{\partial(e^{yz})}{\partial y} - \frac{\partial(\ln(xy))}{\partial z} = z + ze^{yz}$$

**The Curl** of a vector field $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ it is another vector defined as the following determinant:

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)k$$

**Example 8:** Find $curl\ F$ if $F(x,y,z) = xyi + yzj + xzk$ at $(-1,-3,-2)$

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$= \left(\frac{\partial(xz)}{\partial y} - \frac{\partial(yz)}{\partial z}\right)i - \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(xy)}{\partial z}\right)j + \left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)k$$

$$= -yi - zj - xk$$

$$curl\ F\ \Big|_{\text{at }(-1,-3,-2)} = 3i + 2j + k$$

## Exercises

1. Let $u = x\cos y\,, v = x\sin y$. Find Jacobian matrix and its determinant at $(1, \pi/4)$.
2. Let $u = xe^y, v = ye^z,\ \ w = ze^x$. Find Jacobian matrix and its determinant.
3. Find the determinant of the Hessian matrix for the functions

   $(a)\ \ f(x, y) = x^2y + xy^2$.

   $(b)\ \ f(x, y, z) = x^2\sin(yz)$
4. If $f(x, y, z) = x^3y^2z$, then find

   $(a)\quad \nabla f \quad \text{at} \quad (-1,2,-2)$

   $(b)\quad \nabla^2 f \ \ \text{at} \quad (1,-3,2)$
5. If $F(x, y, z) = yze^{xy}i + xze^{xy}j + (e^{xy} + 3\cos 3z)k$, then find

   $(a)\quad div\,F \ \ \text{at} \quad \left(0, \sqrt{6}, \pi/6\right)$

   $(b)\quad curl\,F$