Differential Equations

Introduction

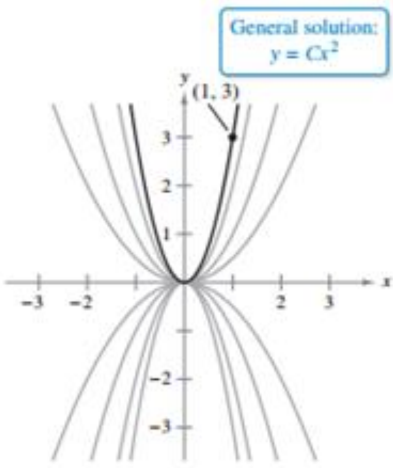
A differential equation (DE) is an equation that involves one or more derivatives. An ordinary differential equation (ODE) is an equation for a function which depends on one independent variable which involves the independent variable, the function, and derivatives of the function. For example: $y'' - 2y'' + y' = 2e^x$, is an ODE, of order two. A partial differential equation (PDE) is an equation for a function which depends on more than one independent variable which involves the independent variables, the function, and partial derivatives of the function. For example: $u_{xy} = 6xy$, is a PDE, of order two.

General Solution and Particular Solution

A particular solution of a DE is any solution that is obtained by assigning specific values to the arbitrary constants in the general solution.

Geometrically, the general solution of a DE represents a family of curves known as solution curves. For instance, the general solution of the differential equation $xy' - 2y = 0$ is

$y = Cx^2$ (General solution)



Solution Curves for $xy' - 2y = 0$

The figure above shows several solution curves corresponding to different values of C. Particular solutions of a differential equation are obtained from initial conditions placed on the unknown function and its derivatives. For instance, suppose you want to find the particular solution whose graph passes through the point $(1,3)$.

This initial condition can be written as $y = 3$ when $x = 1$ (Initial condition)

Substituting these values into the general solution produces $3 = C(1)^2$ which implies that $C = 3$. So, the particular solution is $y = 3x^2$ (Particular solution).

Direct Integration: An ODE of the following form:

$$\frac{d^n y}{dx^n} = f(x)$$

can be integrated directly by finding antiderivatives n-times.

Example 1: Find the general solution of ODE $y' = 2x + 5.$

Solution:

$$\frac{dy}{dx} = 2x + 5 \Rightarrow \quad y = \int (2x+5)dx$$

$$y = x^2 + 5x + C \qquad \text{(General solution)}$$

Example 2: Find the general solution of ODE $y'' = -\sin 4x.$

Solution: By performing the integration process twice for both sides:

$$y' = \int -\sin 4x \, dx = 2\cos 4x + C$$

$$y = \int (2\cos 4x + C)dx = \sin 4x + Cx + K$$

Example 3: Solve ODE $y' = \sec^2 x$ with $y(\pi/4) = 3.$

Solution: $y = \tan x + C$ (General solution)

$$y(\pi/4) = 3 \quad \Rightarrow \quad 3 = \tan(\pi/4) + C$$

$$3 = 1 + C \Rightarrow \quad C = 2$$

$$y = \tan x + 2 \qquad \text{(Particular solution)}$$

Example 4: Solve ODE $y'' = 6x$ with $y'(0) = 2$ and $y(1) = 3.$

Solution: $y' = \int 6x \, dx = 3x^2 + C$

$$y'(0) = 2 \quad \Rightarrow \quad 2 = 3 \times 0 + C \quad \Rightarrow \quad C = 2$$

$$y' = 3x^2 + 2 \Rightarrow \quad y = \int (3x^2 + 2)\, dx = x^3 + 2x + K$$

$$y(1) = 3 \quad \Rightarrow \quad 3 = (1)^3 + 2(1) + K \quad \Rightarrow \quad K = 0$$

$$y = x^3 + 2x \quad \text{(Particular solution)}$$

H.W

1. Solve $y' = \tan x$ with $y(\pi/4) = 3.$
2. Solve $y'' = -x\sin x$ with $y'(0) = -3,\ y(0) = 1.$
3. Solve $y'' = \sin x + 2$ with $y'(0) = 1$, $y(0) = 3.$