6.1. Binomial Distribution

We consider repeated and independent trials of an experiment with two outcomes; we call one of the outcomes *success* and the other outcome *failure*. Let p be the probability of success, so that $q = 1 - p$ is the probability of failure. If we are interested in the number of successes and not in the order in which they occur, then the following theorem applies.

Theorem

The probability of exactly k successes in n repeated trials is denoted and given by

$$b(k; n, p) = \binom{n}{k} p^k q^{n-k}$$

Here $\binom{n}{k}$ is the binomial coefficient ( ). Observe that the probability of no successes is q^n, and therefore the probability of at least one success is $1 - q^n$.

Example 1: A fair coin is tossed 6 times or, equivalently, six fair coins are tossed; call heads a success. Then $n = 6$ and $p = q = \frac{1}{2}$.

(i) The probability that exactly two heads occur (i.e. $k = 2$) is

$$b(2; 6, \tfrac{1}{2}) = \binom{6}{2} (\tfrac{1}{2})^2 (\tfrac{1}{2})^4 = \tfrac{15}{64}$$

(ii) The probability of getting at least four heads (i.e. $k = 4$, 5 or 6) is

$$b(4; 6, \tfrac{1}{2}) + b(5; 6, \tfrac{1}{2}) + b(6; 6, \tfrac{1}{2}) = \binom{6}{4} (\tfrac{1}{2})^4 (\tfrac{1}{2})^2 + \binom{6}{5} (\tfrac{1}{2})^5 (\tfrac{1}{2}) + \binom{6}{6} (\tfrac{1}{2})^6$$

$$= \tfrac{15}{64} + \tfrac{6}{64} + \tfrac{1}{64} = \tfrac{11}{32}$$

(iii) The probability of no heads (i.e. all failures) is $q^6 = (\frac{1}{2})^6 = \frac{1}{64}$, and so the probability of at least one head is $1 - q^6 = 1 - \frac{1}{64} = \frac{63}{64}$.

Example 2: A fair die is tossed 7 times; call a toss a success if a 5 or a 6 appears. Then $n = 7$, $p = P(\{5,6\}) = \frac{1}{3}$ and $q = 1 - p = \frac{2}{3}$.

(i) The probability that a 5 or a 6 occurs exactly 3 times (i.e. $k = 3$) is

$$b(3; 7, \tfrac{1}{3}) = \binom{7}{3}(\tfrac{1}{3})^3(\tfrac{2}{3})^4 = \frac{560}{2187}$$

(ii) The probability that a 5 or a 6 never occurs (i.e. all failures) is $q^7 = (\frac{2}{3})^7 = \frac{128}{2187}$; hence the probability that a 5 or a 6 occurs at least once is $1 - q^7 = \frac{2059}{2187}$.

If we regard n and p as constant, then the above function $P(k) = b(k; n, p)$ is a discrete probability distribution:

k	0	1	2	$\cdots$	n
$P(k)$	q^n	$\binom{n}{1} q^{n-1} p$	$\binom{n}{2} q^{n-2} p^2$	$\cdots$	p^n

It is called the *binomial distribution* since for $k = 0, 1, 2, \ldots, n$ it corresponds to the successive terms of the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots + p^n$$

This distribution is also called the Bernoulli distribution, and independent trials with two outcomes are called Bernoulli trials.

Properties of this distribution follow:

Theorem :

Binomial distribution	
Mean	$\mu = np$
Variance	$\sigma^2 = npq$
Standard deviation	$\sigma = \sqrt{npq}$

Example 3: A fair die is tossed 180 times. The expected number of sixes is $\mu = np = 180 \cdot \frac{1}{6} = 30$. The standard deviation is $\sigma = \sqrt{npq} = \sqrt{180 \cdot \frac{1}{6} \cdot \frac{5}{6}} = 5$.

1. **Example**

Team A has probability $\frac{2}{3}$ of winning whenever it plays. If A plays 4 games, find the probability that A wins (i) exactly 2 games, (ii) at least 1 game, (iii) more than half of the games.

Here $n = 4$, $p = \frac{2}{3}$ and $q = 1 - p = \frac{1}{3}$.

(i) $P(2 \text{ wins}) = b(2; 4, \frac{2}{3}) = \binom{4}{2}(\frac{2}{3})^2(\frac{1}{3})^2 = \frac{8}{27}$.

(ii) Here $q^4 = (\frac{1}{3})^4 = \frac{1}{81}$ is the probability that A loses all four games. Then $1 - q^4 = \frac{80}{81}$ is the probability of winning at least one game.

(iii) A wins more than half the games if A wins 3 or 4 games. Hence the required probability is

$$P(3 \text{ wins}) + P(4 \text{ wins}) = \binom{4}{3}(\frac{2}{3})^3(\frac{1}{3}) + \binom{4}{4}(\frac{2}{3})^4 = \frac{32}{81} + \frac{16}{81} = \frac{16}{27}$$

2.

A family has 6 children. Find the probability P that there are (i) 3 boys and 3 girls, (ii) fewer boys than girls. Assume that the probability of any particular child being a boy is $\frac{1}{2}$.

Here $n = 6$ and $p = q = \frac{1}{2}$.

(i) $P = P(3 \text{ boys}) = \binom{6}{3}(\frac{1}{2})^3(\frac{1}{2})^3 = \frac{20}{64} = \frac{5}{16}$.

(ii) There are fewer boys than girls if there are 0, 1 or 2 boys. Hence

$$P = P(0 \text{ boys}) + P(1 \text{ boy}) + P(2 \text{ boys}) = (\frac{1}{2})^6 + \binom{6}{1}(\frac{1}{2})(\frac{1}{2})^5 + \binom{6}{2}(\frac{1}{2})^2(\frac{1}{2})^4 = \frac{11}{32}$$

3.

Determine the expected number of boys in a family with 8 children, assuming the sex distribution to be equally probable. What is the probability that the expected number of boys does occur?

The expected number of boys is $E = np = 8 \cdot \frac{1}{2} = 4$. The probability that the family has four boys is

$$b(4; 8, \tfrac{1}{2}) = \binom{8}{4}(\tfrac{1}{2})^4(\tfrac{1}{2})^4 = \frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}(\tfrac{1}{2})^8 = \frac{70}{256} = .27$$

4.

The probability is 0.02 that an item produced by a factory is defective. A shipment of 10,000 items is sent to its warehouse. Find the expected number E of defective items and the standard deviation σ.

$E = np = (10{,}000)(0.02) = 200.$

$\sigma = \sqrt{npq} = \sqrt{(10{,}000)(0.02)(0.98)} = \sqrt{196} = 14.$

6.2. Poisson Distribution

The Poisson distribution is defined as follows:

$$p(k; \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots$$

where $\lambda > 0$ is some constant. This countably infinite distribution appears in many natural phenomena, such as the number of telephone calls per minute at some switchboard, the number of misprints per page in a large text, and the number of α particles emitted by a radioactive substance. Diagrams of the Poisson distribution for various values of λ follow.

Properties of the Poisson distribution follow:

Theorem :

Poisson distribution	
Mean	$\mu = \lambda$
Variance	$\sigma^2 = \lambda$
Standard deviation	$\sigma = \sqrt{\lambda}$

Although the Poisson distribution is of independent interest, it also provides us with a close approximation of the binomial distribution for small k provided that p is small and $\lambda = np$. This is indicated in the following table.

k	0	1	2	3	4	5
Binomial	.366	.370	.185	.0610	.0149	.0029
Poisson	.368	.368	.184	.0613	.0153	.00307

Comparison of Binomial and Poisson distributions with $n = 100$, $p = 1/100$ and $\lambda = np = 1$

Example

Suppose 2% of the items made by a factory are defective. Find the probability P that there are 3 defective items in a sample of 100 items.

The binomial distribution with $n = 100$ and $p = .02$ applies. However, since p is small, we use the Poisson approximation with $\lambda = np = 2$. Thus

$$P = p(3; 2) = \frac{2^3 e^{-2}}{3!} = 8(.135)/6 = .180$$

<Best Regards>