

**Parallel-Axis Theorem:**

$$I_{\bar{A}\bar{A}} = \int y^2 dA$$

C: centroid of the area  
&  $\bar{B}\bar{B}$  :centroidal axis

$$\bar{B}\bar{B} // \bar{A}\bar{A}$$

$$I_{\bar{A}\bar{A}} = \int (\bar{y} + d)^2 dA$$

$$I_{\bar{A}\bar{A}} = \int (\bar{y})^2 dA + 2d \int \bar{y} dA + d^2 \int dA$$

$$\int (\bar{y})^2 dA = I_{\bar{B}\bar{B}} = \text{moment of inertia about the centroidal axis}$$

$$\int dA = \text{Area}$$

$$\int \bar{y} dA = 1^{\text{st}} \text{ moment of area about } \bar{B}\bar{B}$$

$$\int \bar{y} dA = \text{zero as } \bar{B}\bar{B} \text{ is the centroidal axis}$$

$$I = \bar{I} + Ad^2$$

$\bar{I}$  : about the centroidal axis

$I$ : about any axis // to the centroidal axis with a distance  $d$

Put  $k^2 A$  for  $I$

$$\& \bar{I} = (\bar{k})^2 A$$

$$k^2 = (\bar{k})^2 + d^2$$

also

$$J_o = J_c + Ad^2$$





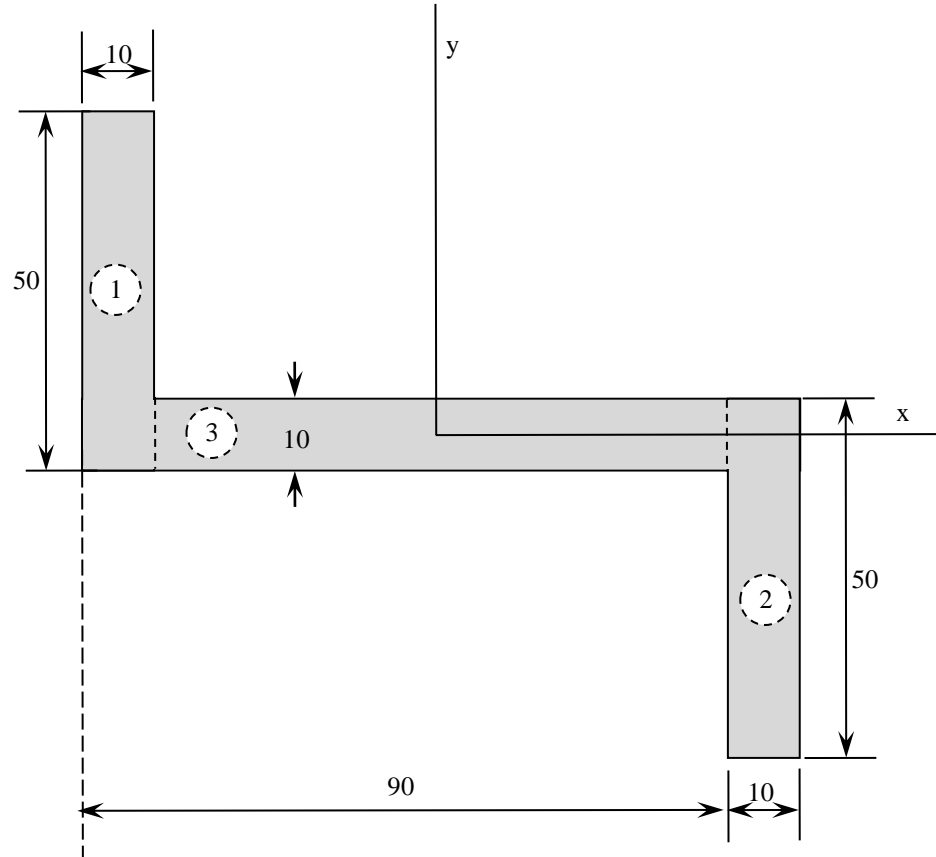
**Moment of Inertia of Composite Area:**

Moments of Inertia of common geometric shapes

Shape		Moments of inertia
Rectangle		$\bar{I}_{x'} = \frac{1}{12}bh^3$ $\bar{I}_{y'} = \frac{1}{12}b^3h$ $I_x = \frac{1}{3}bh^3$ $I_y = \frac{1}{3}b^3h$ $J_C = \frac{1}{12}bh(b^2 + h^2)$
Triangle		$\bar{I}_{x'} = \frac{1}{35}bh^3$ $I_x = \frac{1}{12}bh^3$
circle		$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$ $J_o = \frac{1}{2}\pi r^4$
Semicircle		$\bar{I}_x = \bar{I}_y = \frac{1}{8}\pi r^4$ $J_o = \frac{1}{4}\pi r^4$
Quarter Circle		$\bar{I}_x = \bar{I}_y = \frac{1}{16}\pi r^4$ $J_o = \frac{1}{8}\pi r^4$
Ellipse		$\bar{I}_x = \frac{1}{4}\pi ab^3$ $\bar{I}_y = \frac{1}{4}\pi a^3b$ $J_o = \frac{1}{4}\pi ab(a^2 + b^2)$



**Ex:** Find  $I_{xx}$  &  $k_x$



$$\bar{I}_1 = \frac{b_1 h_1^3}{12} = \frac{10 \cdot 50^3}{12}$$

$$\bar{I}_3 = \frac{b_3 h_3^3}{12} = \frac{80 \cdot 10^3}{12}$$

$$I_{xx} = \bar{I}_3 + 2[\bar{I}_1 + Ad^2]$$

$$I_{xx} = \frac{80 \cdot 10^3}{12} + 2 \left[ \frac{10 \cdot 50^3}{12} + (50 \cdot 10)(25 - 5)^2 \right]$$

$$I_{xx} = 61.5 \cdot 10^4 \text{ mm}^4$$

Or:

$$\text{Part(1)} : I_x = \bar{I}_x + Ad^2 = \frac{b_1 h_1^3}{12} + Ad^2 = \frac{10 \cdot 40^3}{12} + (40 \cdot 10)(20 + 5)^2 = 303333.333 \text{ mm}^4$$

$$\text{Part(2)} : I_x = \bar{I}_x + Ad^2 = \frac{b_2 h_2^3}{12} + Ad^2 = \frac{10 \cdot 40^3}{12} + (40 \cdot 10)(20 + 5)^2 = 303333.333 \text{ mm}^4$$

$$\text{Part(3)} : I_x = \bar{I}_x + Ad^2 = \frac{b_3 h_3^3}{12} = \frac{100 \cdot 10^3}{12} = 8333.333 \text{ mm}^4$$

$$I_{xx} = \Sigma I_{xx} = I_1 + I_2 + I_3 = 303333.333 + 303333.333 + 8333.333 = 61.5 \text{ mm}^4$$

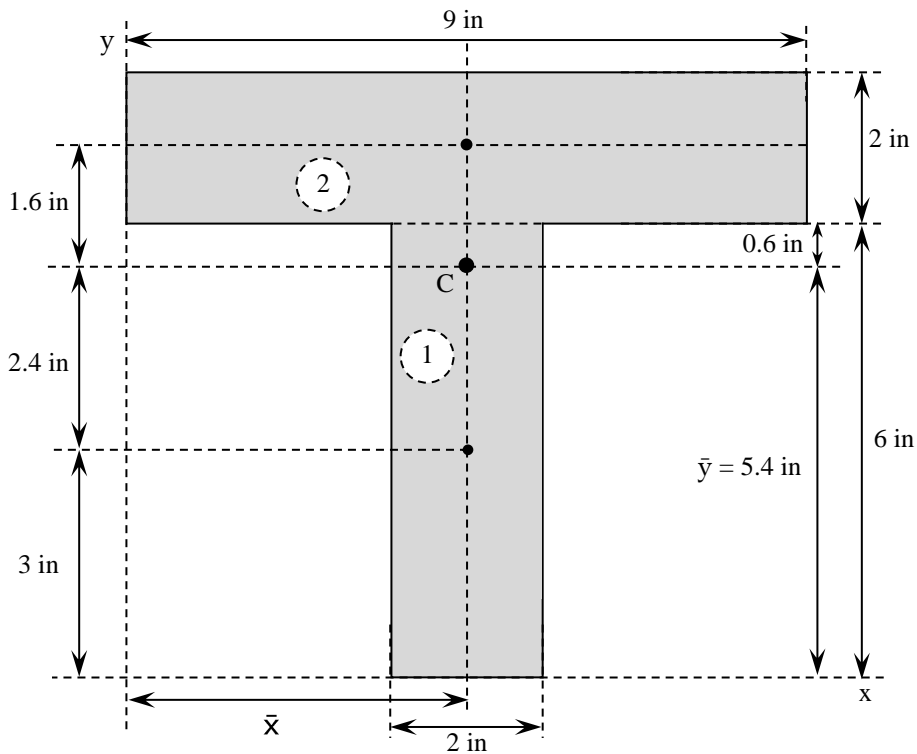
$$A = 50 \cdot 10 + 50 \cdot 10 + 80 \cdot 10 = 1800 \text{ mm}^2$$



$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{61.5 * 10^4}{1800}} = 18.484mm$$

**9.28**

Determine the centroidal polar moment of inertia of the area shown.



Component	A	$\bar{x}$	$\bar{y}$	$A \bar{x}$	$A \bar{y}$
Rectangle 1	$2 * 6 = 12$	4.5	3	54	36
Rectangle 2	$9 * 2 = 18$	4.5	7	81	126
	$\Sigma A = 30$			$\Sigma A \bar{x} = 135$	$\Sigma A \bar{y} = 162$

$$\bar{X} = \frac{\Sigma \bar{x}A}{A} = \frac{135}{30} = 4.5in$$

$$\bar{Y} = \frac{\Sigma \bar{y}A}{A} = \frac{162}{30} = 5.4in$$

$$Part(1) : I_{x_1} = \bar{I}_x + Ad^2 = \frac{b_1 h_1^3}{12} + Ad^2 = \frac{2 * (6)^3}{12} + (2 * 6)(2.4)^2 = 36 + 69.12$$

$$I_{x_1} = 105.12in^4$$



$$\text{Part(2)} : I_{x_2} = \bar{I}_x + Ad^2 = \frac{b_2 h_2^3}{12} + Ad^2 = \frac{9 * (2)^3}{12} + (2 * 9)(1.6)^2 = 6 + 46.08$$

$$I_{x_1} = 52.08 \text{ in}^4$$

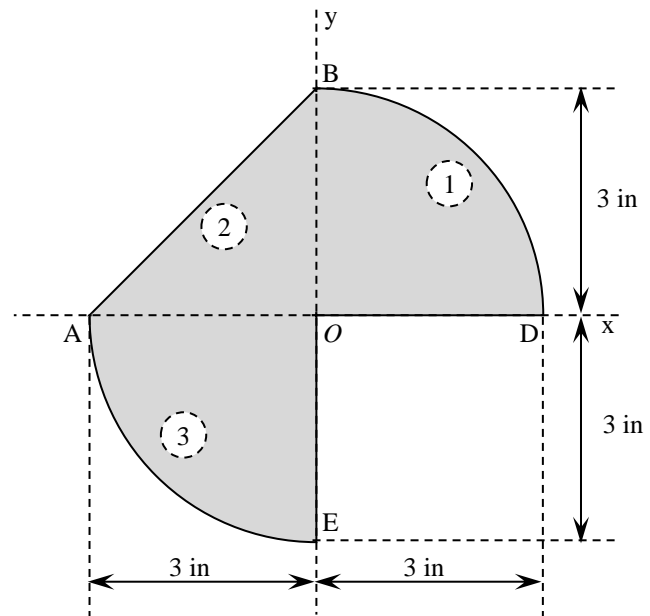
$$I_{x'} = I_{x_1} + I_{x_2} = 105.12 + 52.08 = 157.2 \text{ in}^4$$

$$I_{y'} = I_{y_1} + I_{y_2} = \frac{b_1^3 h_1}{12} + \frac{b_2^3 h}{12} = \frac{(2)^3 * 6}{12} + \frac{(9)^3 * 2}{12} = 4 + 121.5 = 125.5 \text{ in}^4$$

$$J_c = I_{x'} + I_{y'} = 157.2 + 125.5 = 282.7 \text{ in}^4$$

**9.30**

Determine the polar moment of inertia of the area shown with respect to (a) point O , (b) the centroid of the area .

**(a)**

$$I_x = I_{x_1} + I_{x_2} + I_{x_3}$$

$$I_x = \frac{1}{16} \pi r^4 + \frac{1}{12} b h^3 + \frac{1}{16} \pi r^4$$

$$I_x = \frac{1}{16} \pi * (3)^4 + \frac{1}{12} * 3 * (3)^3 + \frac{1}{16} \pi * (3)^4$$

$$I_x = 38.5586 \text{ in}^4$$

$$I_y = I_{y_1} + I_{y_2} + I_{y_3}$$

$$I_y = \frac{1}{16} \pi r^4 + \frac{1}{12} b^3 h + \frac{1}{16} \pi r^4$$

$$I_y = \frac{1}{16} \pi * (3)^4 + \frac{1}{12} * (3)^3 * 3 + \frac{1}{16} \pi * (3)^4$$

$$I_y = 38.5586 \text{ in}^4$$

$$J_o = I_x + I_y = 38.5586 + 38.5586 = 77.11 \text{ in}^4$$

**(b)**

$$\text{Part(1)} : \bar{x} = \frac{+4r}{3\pi} = \frac{+4 * 3}{3\pi} = +1.273 \text{ in}; \bar{y} = \frac{+4r}{3\pi} = \frac{+4 * 3}{3\pi} = +1.273 \text{ in}$$



Part(2) :  $\bar{x} = -\frac{b}{3} = \frac{-3}{3} = -1in; \bar{y} = \frac{h}{3} = \frac{3}{3} = +1in$

Part(3) :  $\bar{x} = \frac{-4r}{3\pi} = \frac{-4*3}{3\pi} = -1.273in.; \bar{y} = \frac{-4r}{3\pi} = \frac{-4*3}{3\pi} = -1.273in$

Component	A	$\bar{x}$	$\bar{y}$	A $\bar{x}$	A $\bar{y}$
Quarter circle 1	$= \frac{\pi r^2}{4} = \frac{\pi * 3^2}{4}$ $= 7.068$	$= \frac{4r}{3\pi} = \frac{4*3}{3\pi}$ $= 1.2732$	$= \frac{4r}{3\pi} = \frac{4*3}{3\pi}$ $= 1.2732$	8.999	8.999
triangle 2	$= \frac{bh}{2} = \frac{3*3}{2}$ $= 4.5$	$= \frac{-b}{3} = \frac{-3}{3}$ $= -1$	$= \frac{h}{3} = \frac{3}{3}$ $= 1$	-4.5	4.5
Quarter circle 3	$= \frac{\pi r^2}{4} = \frac{\pi * 3^2}{4}$ $= 7.068$	$= \frac{-4r}{3\pi} = \frac{-4*3}{3\pi}$ $= -1.2732$	$= \frac{-4r}{3\pi} = \frac{-4*3}{3\pi}$ $= -1.2732$	- 8.999	- 8.999
----	$\Sigma A = 18.636$	----	----	$\Sigma A \bar{x} = -4.5$	$\Sigma A \bar{y} = 4.5$

$\bar{x} = \frac{\sum \bar{x}A}{A} = \frac{-4.5}{18.636} = -0.242468in$

$\bar{y} = \frac{\sum \bar{y}A}{A} = \frac{4.5}{18.636} = +0.241468in$

$I_{x_1} = I_{x'_1} + A_1(\bar{y}_1)^2$

$I_{x'_1} = \frac{1}{16} \pi r^4 - \frac{\pi r^2}{4} * (1.273)^2$

$I_{x'_1} = \frac{1}{16} \pi * 3^4 - \frac{\pi 3^2}{4} * (1.273)^2$

$I_{x'_1} = 4.45in^4$

$I_{cx_1} = I_{x'_1} + A_1(\bar{y}_1 - \bar{y})^2$

$I_{cx_1} = 4.45 + \frac{\pi r^2}{4} * (1.273 - 0.2414)^2$

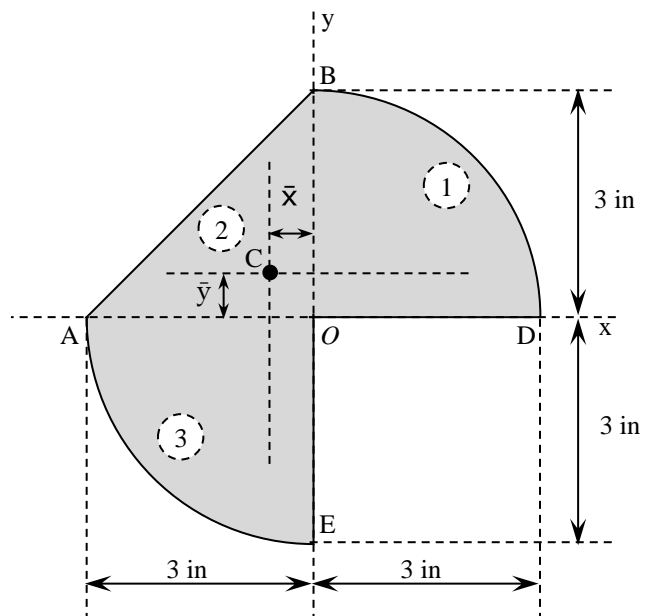
$I_{cx_1} = 4.45 + \frac{\pi 3^2}{4} * (1.0316)^2$

$I_{cx_1} = 11.9723in^4$

$I_{x_2} = I_{x'_2} + A_2(\bar{y}_2)^2$

$I_{x'_2} = \frac{1}{12} bh^3 - \frac{bh}{2} * (1)^2 = \frac{1*3*3^3}{12} - \frac{3*3}{2} * (1)^2$

$I_{x'_2} = 2.25in^4$





$$I_{cx_2} = I_{x'_2} + A_2(\bar{y}_2 - \bar{y})^2$$

$$I_{cx_2} = 2.25 + \frac{bh}{2} * (1 - 0.2414)^2 = 2.25 + \frac{3*3}{2} * (0.7586)^2$$

$$I_{cx_2} = 4.8396 \text{ in}^4$$

$$I_{x_3} = I_{x'_3} + A_3(\bar{y}_3)^2$$

$$I_{x'_3} = \frac{1}{16} \pi r^4 - \frac{\pi r^2}{4} * (1.273)^2 = \frac{1}{16} \pi * 3^4 - \frac{\pi 3^2}{4} * (1.273)^2$$

$$I_{x'_3} = 4.45 \text{ in}^4$$

$$I_{cx_3} = I_{x'_3} + A_3(\bar{y}_3 + \bar{y})^2$$

$$I_{cx_3} = 4.45 + \frac{\pi r^2}{4} * (1.273 + 0.2414)^2 = 4.45 + \frac{\pi 3^2}{4} * (1.5144)^2$$

$$I_{cx_3} = 20.6611 \text{ in}^4$$

$$I_{cx} = I_{cx_1} + I_{cx_2} + I_{cx_3}$$

$$I_{cx} = 11.9723 + 4.839 + 20.6611$$

$$I_{cx} = 37.473 \text{ in}^4$$

$$I_{y_1} = I_{y'_1} + A_1(\bar{x}_1)^2$$

$$I_{y'_1} = \frac{1}{16} \pi r^4 - \frac{\pi r^2}{4} * (1.273)^2 = \frac{1}{16} \pi * 3^4 - \frac{\pi 3^2}{4} * (1.273)^2$$

$$I_{y'_1} = 4.45 \text{ in}^4$$

$$I_{cy_1} = I_{y'_1} + A_1(\bar{x}_1 + \bar{x})^2$$

$$I_{cy_1} = 4.45 + \frac{\pi r^2}{4} * (1.273 + 0.2414)^2 = 4.45 + \frac{\pi 3^2}{4} * (1.5144)^2$$

$$I_{cy_1} = 20.6611 \text{ in}^4$$

$$I_{y_2} = I_{y'_2} + A_2(\bar{x}_2)^2$$

$$I_{y'_2} = \frac{1}{12} b^3 h - \frac{bh}{2} * (1)^2 = \frac{1*3^3*3}{12} - \frac{3*3}{2} * (1)^2$$

$$I_{y'_2} = 2.25 \text{ in}^4$$

$$I_{cy_2} = I_{y'_2} + A_2(\bar{x}_2 - \bar{x})^2$$

$$I_{cy_2} = 2.25 + \frac{bh}{2} * (1 - 0.2414)^2 = 2.25 + \frac{3*3}{2} * (0.7586)^2$$

$$I_{cy_2} = 4.8396 \text{ in}^4$$

$$I_{y_3} = I_{y'_3} + A_3(\bar{x}_3)^2$$

$$I_{y'_3} = \frac{1}{16} \pi r^4 - \frac{\pi r^2}{4} * (1.273)^2 = \frac{1}{16} \pi * 3^4 - \frac{\pi 3^2}{4} * (1.273)^2$$

$$I_{y'_3} = 4.45 \text{ in}^4$$



$$I_{cy_3} = I_{xy_3'} + A_3(\bar{x}_3 - \bar{x})^2$$

$$I_{cy_3} = 4.45 + \frac{\pi r^2}{4} * (1.273 - 0.2414)^2 = 4.45 + \frac{\pi 3^2}{4} * (1.0316)^2$$

$$I_{cy_3} = 11.9723in^4$$

$$I_{cy} = I_{cy_1} + I_{cy_2} + I_{cy_3}$$

$$I_{cy} = 20.6611 + 4.839 + 11.9723$$

$$I_{cy} = 37.473in^4$$

$$J_c = I_{cx} + I_{cy}$$

$$J_c = 37.473 + 37.573$$

$$J_c = 74.946in^4$$