Derivatives of Trigonometric Functions

$360° = 2\pi$ radians	$1 \text{ radians} = \dfrac{180°}{\pi} = 57°17'44.8'' = 57 + \dfrac{17}{60} + \dfrac{44.8}{60 \times 60}$
π = 180 ° (sin π; cos π, sec π, csc π, cot π, and tan π)	
$\pi = \frac{22}{7}$ in real no. (2π; e^{π}, ln π, … etc)	

1	$\frac{d}{dx}\sin x = \cos x \ dx$	$\frac{d}{dx}\sin u = \cos u \ \frac{du}{dx}$
2	$\frac{d}{dx}\cos x = -\sin x \ dx$	$\frac{d}{dx}\cos u = -\sin u \ \frac{du}{dx}$
3	$\frac{d}{dx}\tan x = \sec^2 x \ dx$	$\frac{d}{dx}\tan u = \sec^2 u \ \frac{du}{dx}$
4	$\frac{d}{dx}\sec x = \sec x \ \tan x \, dx$	$\frac{d}{dx}\sec u = \sec u \ \tan u \frac{du}{dx}$
5	$\frac{d}{dx}\csc x = -\csc x \ \cot x \, dx$	$\frac{d}{dx}\csc u = -\csc u \ \cot u \frac{du}{dx}$
6	$\frac{d}{dx}\cot x = -\csc^2 x \ dx$	$\frac{d}{dx}\cot u = -\csc^2 u \ \frac{du}{dx}$

Applications of Derivatives

1. Increases and decreases of function:

(a) f(x) increases if $f'(x) > 0$;

(b) f(x) decreases if $f'(x) < 0$;

2. Concave up, concave down and inflection points:

(a) Concave down $y'' < 0$;

(b) Concave up $y'' > 0$;

Example 1: Graph the curve $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$

Solution: (1) $y' = \frac{dy}{dx} = \frac{1}{6}(3x^2 - 12x + 9)$

$$y'' = \frac{d^2y}{dx^2} = \frac{1}{6}(6x - 12) = x - 2$$

(2) For maximum and minimum values of y:

$$y' = 0 \;;;;\; 0 = \frac{1}{6}(3x^2 - 12x + 9) \;;;;\; (x^2 - 4x + 3) = 0$$

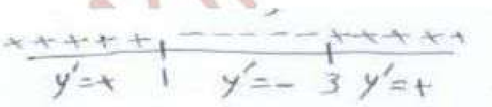
$$(x-1)(x-3) = 0 \;;;\; x = 1 \;\therefore y = \frac{5}{3} \;;;\left(1, \frac{5}{3}\right)$$

$$x = 3 \;\therefore y = 1 \;;;(3, 1)$$

$$where \quad y' = (x^2 - 4x + 3)$$

at x =1 (max. point)

at x =3 (min. point)



(3) For concave up and down:

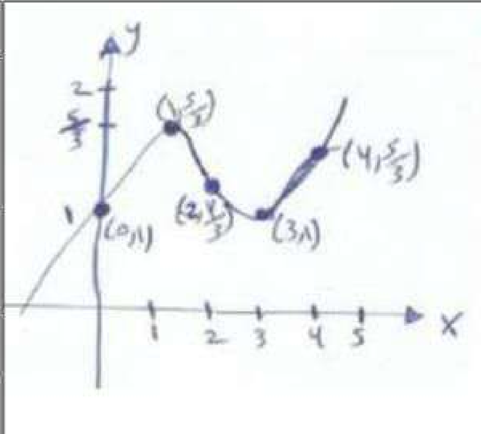
$$y'' = 0 \;;;;\; 0 = x - 2 \;;;;\; x = 2$$

$$\therefore y = \frac{4}{3} \;;;\left(2, \frac{4}{3}\right) \; inflection\ point$$

$$x < 2 \;;;\; y'' = - \quad (concave\ down)$$

$$x > 2 \;;;\; y'' = + \quad (concave\ up)$$

(4) Graph:

x	y		
0	1	For Graph	
1	$\frac{5}{3}$	max point	
2	$\frac{4}{3}$	inflection point	
3	1	min point	
4	$\frac{5}{3}$	For Graph	

Asymptotes:

1. A line y = b is horizontal asymptote:

$$\lim_{x \to \pm\infty} f(x) = b$$

2. A line x = a is vertical asymptote:

$$\lim_{x \to a} f(x) = \pm\infty$$

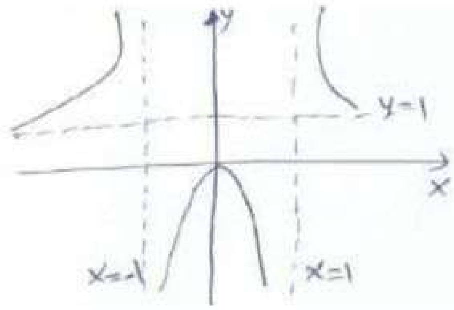
3. If a rational function is a quotient of two polynomials that have no common factor and if degree of numerator in one larger than the degree of denominator as it is in $y = \frac{x^2 - 3}{2x - y}$ then the graph have an oblique asymptote.

Example 1: Find vertical and horizontal asymptote of $y = \frac{x^2}{x^2 - 1}$

Solution:

(1) horizontal asymptote: $\lim_{x\to\infty} \frac{2x}{2x} = 1$;; y = 1 h. a.

(2) Vertical asymptote: $\lim_{x\to a} \frac{x^2}{x^2 - 1} = \pm\infty$;;; $x = \pm\infty$ v.a.



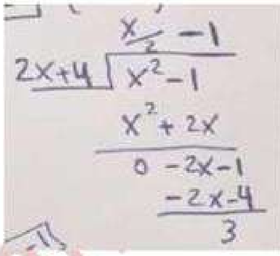
Example 2: Find vertical, horizontal and oblique asymptote of $y = \frac{x^2 - 1}{2x + 4}$

Solution:

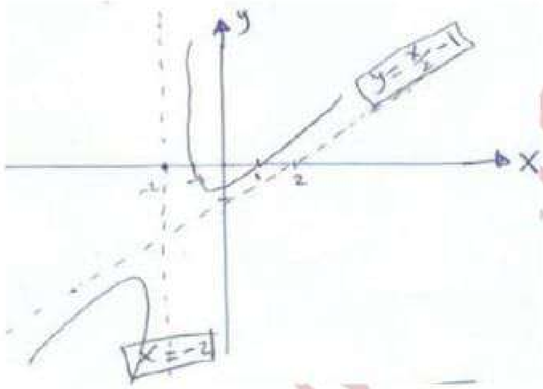
(1) horizontal asymptote: $\lim_{x\to\infty} \frac{x^2-1}{2\,x\,+4} = \frac{\infty}{\infty}$;; $\lim_{x\to\infty} \frac{2x}{2} = \infty$ no. h. a.

(2) vertical asymptote: $\lim_{x\to a} \frac{x^2-1}{2\,x\,+4} = \infty$;; $\lim_{x\to a} x = -2$ v. a.

(3) Oblique asymptote: $y = \frac{x^2-1}{2\,x\,+4} = \frac{x}{2} - 1 + \frac{3}{2\,x\,+4}$



$y = \frac{x}{2} - 1$ O.a.



Example 3: If $y = \frac{x^2+1}{x+1}$ Find:

1. Max. and min point;
2. Concave up and down, inflection points;
3. Asymptotes;
4. Graph of function;
5. Domain and range.

Solution:

$$(1)\ y' = \frac{(x\,+\,1)\times(2x) - (x^2+\,1)\times(1)}{(x\,+\,1)^2} = \frac{x^2+2x-1}{(x\,+\,1)^2}$$

$$y' = 0\ \ ;;;\ \frac{x^2+2x-1}{(x+1)^2} = 0$$

$$x^2+2x-1=0\,;\ x = \frac{-2\pm\sqrt{4+4}}{2}\,;x=-2.41\ \ \rightarrow y = -4.82$$

$$x = 0.41\ \ \rightarrow y = 0.82$$

max. point: $(-2.41, -4.82)$;;; min. point: $(0.41, 0.82)$

$$(2)\ y'' = \frac{(x+1)^2 \times (2x+2) - (x^2+2x-1) \times 2(x+1)}{(x+1)^4}$$

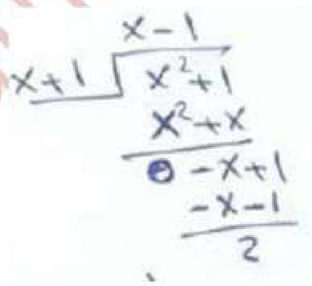
$$y'' = \frac{(x+1) \times (2x+2) - (x^2+2x-1) \times 2}{(x+1)^3} = \frac{2x^2+4x+2-2x^2-4x+2}{(x+1)^3}$$

$$= \frac{4}{(x+1)^3}$$

$$y''=0\ ;;;\ \frac{4}{(x+1)^3} = 0\ \ ;;\ \ 4 \neq 0\ \ (no\ inflection\ point)$$

(3) horizontal asymptote: $\lim_{x\to\infty} \frac{x^2+1}{x+1}$;; $\lim_{x\to\infty} \frac{2x}{1} = \infty$ (no. h. a.)

vertical asymptote: $\lim_{x\to a} \frac{x^2+1}{x+1}$;; $x = -1$ h. a.



Oblique asymptote: $y = \frac{x^2+1}{x+1}$; $y = x - 1 + \frac{2}{x+1}$

$y = x - 1$ O.a.

For $x = -1$;; $y'' = +$ $\rightarrow x > -1$ (concave up)

$y'' = -$ $\rightarrow x < -1$ (concave down)

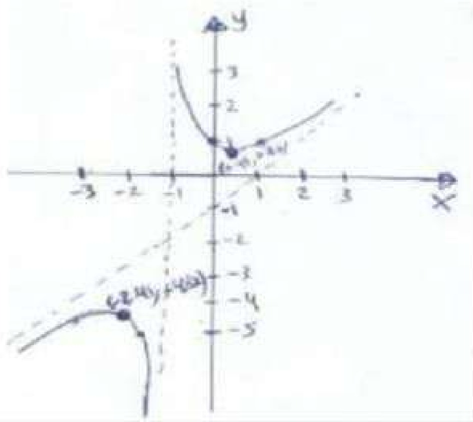
x	y
0	1
0.41	0.82
1	1

x	y
-2	-5
-2.41	-4.82
-3	-5

x	y = x-1 (O.a.)
0	-1
1	0

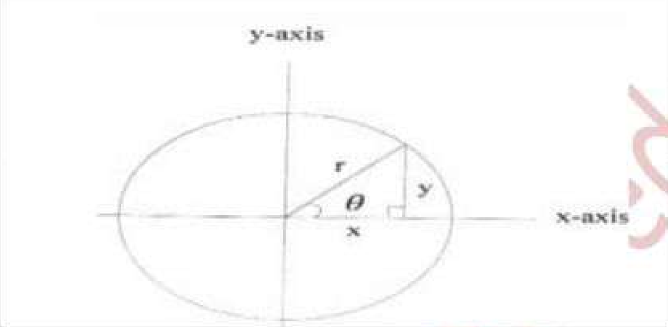
5. D_f: $x \neq -1$

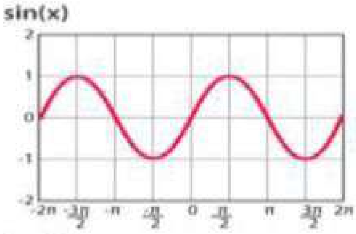
R_f; $0.82 \leq y \cup y \leq -4.82$

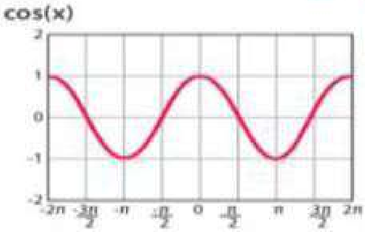


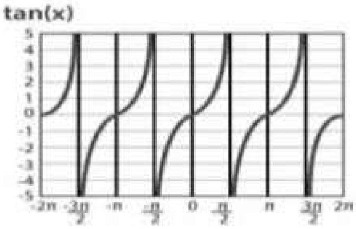
Trigonometric Functions:

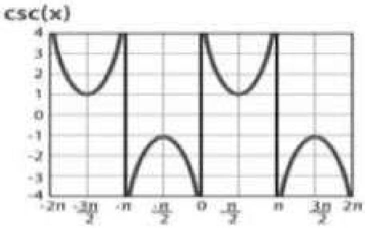
Given a circle of radius (r) and p(x,y) is any point of the circle then:

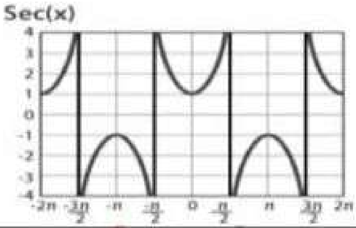
$\sin\theta = \frac{y}{r}$	$\csc\theta = \frac{1}{sin\theta} = \frac{r}{y}$	
$\cos\theta = \frac{x}{r}$	$\sec\theta = \frac{1}{cos\theta} = \frac{r}{x}$	
$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$	$\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$	

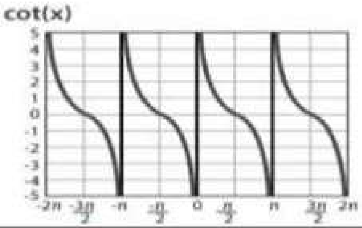


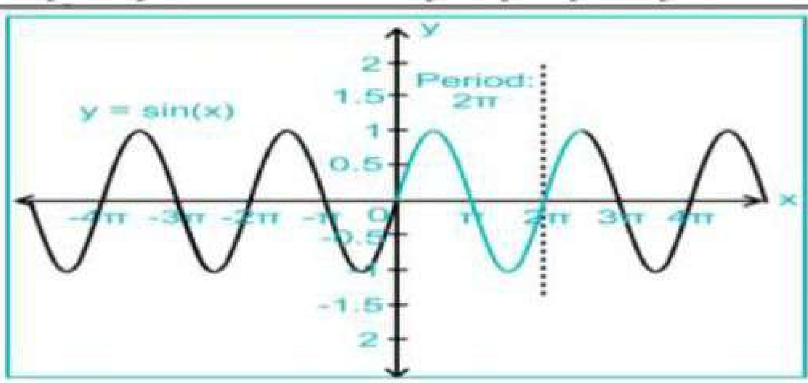




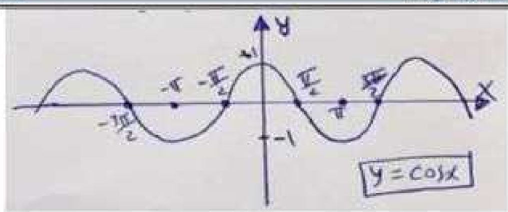




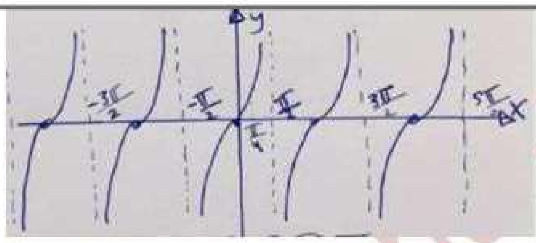


x	y = sin x	
0	0	
$\frac{\pi}{2}$	1	
π	0	
$-\frac{\pi}{2}$	-1	
$-\pi$	0	

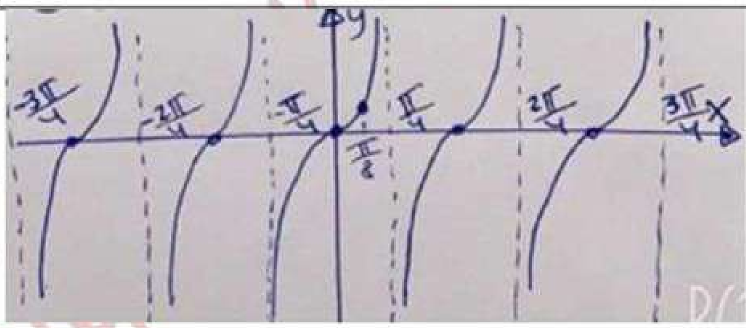
x	y = cos x
0	1
$\frac{\pi}{2}$	0
$-\frac{\pi}{2}$	0
$-\pi$	-1



x	y = tan x
0	0
$\frac{\pi}{2}$	∞
$-\frac{\pi}{2}$	$-\infty$
$\frac{\pi}{4}$	1
$-\frac{\pi}{4}$	-1



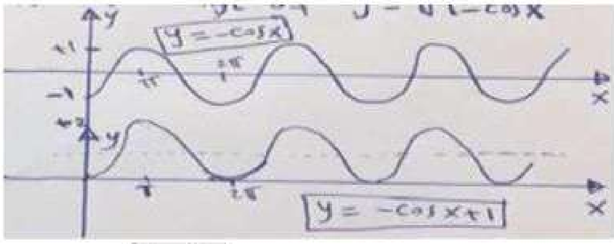
x	y = tan 2x
0	0
$\frac{\pi}{4}$	∞
$\frac{\pi}{2}$	0
$\frac{\pi}{8}$	1



Example: Find the domain and range of the function $y = \sin^2 x$ and graph the function.

Solution: $y = \sqrt{1 - \cos x}$

$y = \sqrt{-\cos x + 1}$



From graph: $Df: \ -\infty \leq x \leq \infty$

$Rf: \ 0 \leq y \leq \sqrt{2}$

Example: Find the domain and range of the function $y = \sqrt{\sin 2x}$.

Solution: For $0 \leq x \leq \frac{\pi}{2}[+]$

For all :Df: $0 \pm \frac{4\pi}{2} \leq x \leq \frac{\pi}{2} \pm \frac{n\pi}{2} \ (n = 0, 2, 4, 6, \dots etc)$

or: $0 \pm n\pi \leq x \leq \frac{\pi}{2} \pm n\pi \ (n = 0, 1, 2, 3, \dots etc)$

$$Rf: \ 0 \leq y \leq 1$$