Lec. 1

تحليلات هندسية

Laplace Transformation

$(\mathcal{L})$

$$F(s) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt = \mathcal{L}[f(t)]$$

let $y = f(t) \Longrightarrow \mathcal{L}[f(t)] = F(s)$

- IF $\boxed{f(t) = a}$ where a = Constant

$$\therefore F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$\therefore F(s) = \int_0^\infty a \cdot e^{-st} \cdot dt = \frac{-a}{s} \int_0^\infty e^{-st}(-s)\, dt$$

$$F(s) = \frac{-a}{s}\left[e^{-st}\right]_0^\infty = \frac{a}{s}$$

$$\boxed{\therefore F(s) = \frac{a}{s}}$$

② IF $f(t) = t^n$

$$\therefore F(s) = \frac{n!}{s^{n+1}}$$

or $\mathcal{L}[f(t)] = \mathcal{L}\, t^n = \frac{n!}{s^{n+1}}$

EX: Find the Laplace transformation (L.T.) of $f(t)$:-

① $f(t) = 3$
② $f(t) = 1$
③ $f(t) = t^4$
④ $f(t) = t^3$

Sol:-

$$\therefore \mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$\therefore$ ① $\mathcal{L}[3] = F(s) = \frac{3}{s}$

② $\mathcal{L}[1] = F(s) = \frac{1}{s}$

③ $\mathcal{L}[t^4] = F(s) = \dfrac{4!}{s^5}$

④ $\mathcal{L}[t^3] = F(s) = \dfrac{3!}{s^4}$

③ $\boxed{\text{IF } f(t) = e^{at}}$

$\boxed{\therefore F(s) = \dfrac{1}{s-a}}$

EX:- Find the laplace Transformation (L.T.) of $f(t)$:-

① $f(t) = e^{3t}$

② $f(t) = e^{-3t}$

③ $f(t) = e^{t}$

Sol :- $\therefore \mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$

$\therefore$ ① $\mathcal{L}[e^{3t}] = F(s) = \dfrac{1}{s-3}$

(2) $\mathcal{L}[e^{-3t}] = F(s) = \frac{1}{s+3}$

(3) $\mathcal{L}[e^{t}] = F(s) = \frac{1}{s-1}$

(4) IF $f(t) = \sin at$

$\therefore F(s) = \frac{a}{s^2+a^2}$

(5) IF $f(t) = \cos at$

$\therefore F(s) = \frac{s}{s^2+a^2}$

(6) IF $f(t) = \sinh at$

$F(s) = \frac{a}{s^2-a^2}$

4 IF $f(t) = \cosh at$

$$\therefore F(s) = \frac{s}{s^2 - a^2}$$

Ex:- Find L.T. of $f(t)$:-

1. $f(t) = \cos 7t$
2. $f(t) = \sin 3t$
3. $f(t) = \sinh 4t$
4. $f(t) = \cosh t$

sol:-

$$\because \mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$\therefore ① \; F(s) = \frac{s}{s^2 + 49} \qquad \text{where } a = 7$$

$$② \; F(s) = \frac{3}{s^2 + 9} \qquad \text{where } a = 3$$

③ $F(s) = \frac{4}{s^2-16}$ where $\{a=4\}$

④ $F(s) = \frac{s}{s^2-1}$ where $\{a=1\}$

The linearity of Laplace Transformation

① If we have $(f(t))$ and $(g(t))$ two function of time and have a Laplace transformation for all $t > 0$ Then :-

$$\mathcal{L}[a f(t) + b g(t)] = a\,\mathcal{L}[f(t)] + b\,\mathcal{L}[g(t)]$$

Ex :- Find L.T. for $[f(t) = 1 + t^3 - 4e^{2t} + 7\cos 6t]$

Sol :-

$$\mathcal{L}[f(t)] = F(s) = \mathcal{L}\,1 + \mathcal{L}\,t^3 - 4\,\mathcal{L}\,e^{2t} + 7\,\mathcal{L}\cos 6t$$

$\therefore F(s) = \frac{1}{s} + \frac{3!}{s^4} - 4\frac{1}{s-2} + 7\frac{s}{s^2+36}$

② Shifting :-

$$\mathcal{L}\left[e^{at} \cdot f(t)\right] = f(s-a)$$

EX1 :- Find L.T. of $\mathcal{L}\left[e^{3t} \cdot \cos 4t\right]$

Sol :-

$f(t) = \cos 4t \quad , \quad a = 3$

$F(s) = \mathcal{L}[f(t)] = \mathcal{L}(\cos 4t) = \frac{s}{s^2+16}$

$\therefore \mathcal{L}\, e^{3t} \cdot \cos 4t = \frac{(s-3)}{(s-3)^2+16}$

{Ex₂}:- Find L.T. of $\left[e^{-4t}(\cos 7t - \sinh 3t)\right]$

{Sol}:-

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[(\cos 7t - \sinh 3t)]$$

$$\therefore F(s) = \mathcal{L}\cos 7t - \mathcal{L}\sinh 3t$$

$$F(s) = \frac{s}{s^2+49} - \frac{3}{s^2-9}$$

$$\therefore \mathcal{L}\, e^{-4t} \cdot f(t) = \frac{(s+4)}{(s+4)^2+49} - \frac{3}{(s+4)^2 - 9}$$

where $(a=-4)$

{Ex₃}:- Find L.T. of $\left[e^{t}\cosh 2t - 3\sin 4t \right]$

{sol}

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}\, e^{t}\cosh 2t - 3\,\mathcal{L}\sin 4t$$

$$\mathcal{L}\, e^{t}\cosh 2t = \mathcal{L}\, e^{t} \cdot f(t)$$

$$\mathcal{L}\cosh 2t = \frac{s}{s^2-4}$$

$$\therefore \mathcal{L}\, e^{t} \cdot \cosh 2t = \frac{(s-1)}{(s-1)^2-4} \quad \text{where } (a=1)$$

$$\therefore F(s) = \frac{(s-1)}{(s-1)^2-4} - 3\,\frac{4}{s^2+16}$$

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③ Derivitive :-

$$\mathcal{L}\left[t^{n} \cdot f(t)\right] = (-1)^{n}\, \frac{d^{n}f}{ds^{n}}$$

Ex₁ :- Find L.T. of $[t \cdot \cos 4t]$ $n=1$

Sol :- $f(t) = \cos 4t$ where $n=1$

$$F(s) = \mathcal{L}[f(t)] = \frac{s}{(s^2+16)}$$

$$\frac{d^{n}f}{ds^{n}} = \frac{(s^2+16)(1)-(s)(2s)}{(s^2+16)^2} = \frac{s^2+16-2s^2}{(s^2+16)^2}$$

$$\therefore \frac{dF}{ds} = \frac{16-s^2}{(s^2+16)^2}$$

$$\therefore \mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \frac{d^nF}{ds^n}$$

$$= (-1)^1 \frac{16-s^2}{(s^2+16)^2} = \frac{s^2-16}{(s^2+16)^2}$$

Ex₂ . Find L.T. of $(t^{\overset{=n}{\boxed{2}}} \cdot \sinh 3t)$

Sol

$f(t) = \sinh 3t$, where $\boxed{n=2}$

$$\therefore F(s) = \mathcal{L}\{f(t)\} = \frac{3}{s^2-9}$$

$$\frac{d^nF}{ds^n} = \frac{d^2F}{ds^2}$$

اولاً نستخرج المشتقة الأولى للدالة F(s) ومن ثم نشتقها مرة ثانية للحصول على $\left(\frac{d^2F}{ds^2}\right)$

$$\frac{dF}{ds} = \frac{(s^2-9)(0)-(3)(2s)}{(s^2-9)^2} = \frac{-6s}{(s^2-9)^2}$$

$$\frac{d^2 f}{ds^2} = \frac{(s^2-9)^2(-6) - (-6s)(2)(s^2-9)(2s)}{(s^2-9)^4}$$

$$= \frac{-6(s^2-9)^2 + 24s^2(s^2-9)}{(s^2-9)^4}$$

$$\mathcal{L}\, t^2 \sinh 3t = (-1)^2 \frac{-6(s^2-9)^2 + 24s^2(s^2-9)}{(s^2-9)^4}$$

$$= \frac{-6(s^2-9)^2 + 24s^2(s^2-9)}{(s^2-9)^4}$$

EX 3 :- Find L.T. of $[\, t \cdot e^{t} \cdot \cos t \,]$

Sol :-

$$\mathcal{L}\, t \cdot \cos t \Longrightarrow f(t) = \cos t$$

$$n = 1$$

$$F(s) = \mathcal{L}\, f(t) = \frac{s}{s^2+1}$$

$$\frac{d^n f}{ds^n} = \frac{df}{ds} = \frac{(s^2+1)(1) - (s)(2s)}{(s^2+1)^2}$$

$$\frac{df}{ds} = \frac{s^2+1-2s^2}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$\therefore \mathcal{L}\, t^n f(t) = (-1)^n \cdot \frac{d^n f}{ds^n}$$

$$= (-1)^1 \frac{1-s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

$$\mathcal{L}\, e^{at} \cdot f(t) = F(s-a)$$

$$\mathcal{L}\, e^{t}(t\cos t) = \frac{(s-1)^2-1}{((s-1)^2+1)^2}$$

Ex4 Find L.T. of $\left(f(t) = t^2 e^{3t}\right)$

sol.

هناك طريقتان لحل هذا المثال

① الطريقة الأولى :

assume $f(t) = t^2$

$$\therefore F(s) = \mathcal{L} f(t) = \frac{n!}{s^{n+1}}$$

$$F(s) = \mathcal{L}(t^2) = \frac{2!}{s^3}$$

$$\mathcal{L}\, e^{at} f(t) = f(s-a)$$

$$\boxed{\mathcal{L}\, e^{3t} \cdot t^2 = \frac{2!}{(s-3)^3}}$$

الطريقة الثانية :-

assume $f(t) = e^{3t}$

$$\mathcal{L}\, e^{at} = \frac{1}{s-a} \Rightarrow \mathcal{L}\, e^{3t} = \frac{1}{s-3}$$

$$\mathcal{L}\, t^n f(t) = (-1)^n \frac{d^n f}{ds^n} \qquad \boxed{n=2}$$

$$\frac{d^2 f}{ds^2} = \frac{d}{ds}\left(\frac{df}{ds}\right)$$

$$\frac{df}{ds} = \frac{(s-3)(0) - (1)(1)}{(s-3)^2} = \frac{-1}{(s-3)^2}$$

$$\frac{d^2 f}{ds^2} = \frac{(s-3)^2(0) - (-1)(2)(s-3)(1)}{(s-3)^4}$$

$$\frac{d^2f}{ds^2} = \frac{2(s-3)}{(s-3)^4} \Longrightarrow \frac{d^2f}{ds^2} = \frac{2}{(s-3)^3}$$

$$\therefore \mathcal{L}\, t^2 e^{3t} = (-1)^2 \frac{2}{(s-3)^3}$$

$$\boxed{\mathcal{L}\, t^2 e^{3t} = \frac{2}{(s-3)^3}}$$