Techniques of Integration:

1.Substitution Method :

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a \tan \theta$, $x = a \sin \theta$, and $x = a \sec \theta$. They come from the reference right triangles in Figure 8.2.

With $x = a \tan \theta$,

$$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta.$$

With $x = a \sin \theta$,

$$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta.$$

With $x = a \sec \theta$,

$$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta.$$

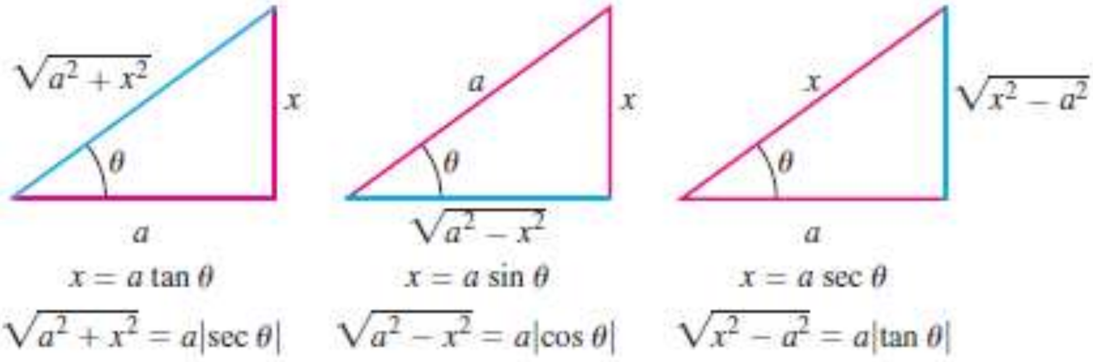

FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a \tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a \sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a \sec\theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1. \end{cases}$$

هذه خطوات لاستخدام هذه الطريقة:

١- اذا المقدار ليس ١ تكون الفرضية حسب الشكل المثلثي *جذر ذالك الثابت
٢- اذا المقدار يتكون من حد ثابت =١ واخر متغير مضروب في مقدار معين فعند ذلك تكون الفرضية حسب الشكل المثلثي مقسوم على جذر ذلك الثابت
٣- يتم استخدام مثلث فيثاغورس القائم الزاوية لزاوية واحدة معينة

Example:

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

Then

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$
$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \qquad \text{From Fig. 8.4}$$
$$= \ln|\sqrt{4+x^2} + x| + C'. \qquad \text{Taking } C' = C - \ln 2$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

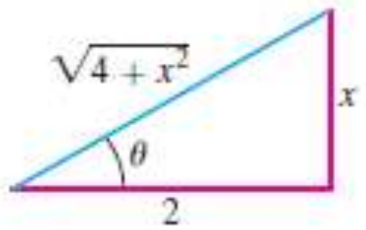

FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Example:

Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

Then

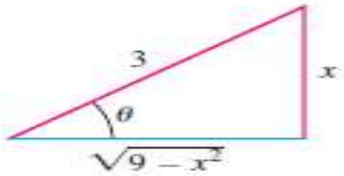

FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

$$\begin{aligned}
\int \frac{x^2\,dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\,d\theta && \cos\theta > 0 \text{ for } -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2} \\
&= 9\int \frac{1-\cos 2\theta}{2}\,d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$

■

Example:

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$

$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$

Fig. 8.6 ■

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

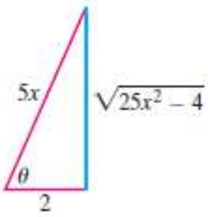

FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

2-By Part Method:

Since

$$\int x\,dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\, dx = \int [f'(x)g(x) + f(x)g'(x)]\, dx$$

EXAMPLE 1 Using Integration by Parts

Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u\, dv = uv - \int v\, du$ with

$$u = x, \qquad dv = \cos x\, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\, dx = x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$$

Let us examine the choices available for u and dv in Example 1.

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

Example:

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x)\, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x)\, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.