2.1. Sample Spaces and Events

An experiment is any action or process whose outcome is subject to uncertainty. Although the word experiment generally suggests a planned or carefully controlled laboratory testing situation, we use it here in a much wider sense. Thus experiments that may be of interest include tossing a coin once or several times, selecting a card or cards from a deck, weighing a loaf of bread, ascertaining the commuting time from home to work on a particular morning, obtaining blood types from a group of individuals, or calling people to conduct a surve.

DEFINITION :The sample space of an experiment, denoted by S, is the set of all possible outcomes of that experiment

DEFINITION: An event is any collection (subset) of outcomes contained in the sample space S. An event is said to be simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

DEFINITION When A and B have no outcomes in common, they are said to be disjoint or mutually exclusive events. Mathematicians write this compactly as $A \cap B = \emptyset$where $\emptyset$denotes the event consisting of no outcomes whatsoever (the "null" or "empty" event).

Example :

1.

Experiment: Toss a die and observe the number that appears on top. Then the sample space consists of the six possible numbers:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event that an even number occurs, B that an odd number occurs and C that a prime number occurs:

$$A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}, \quad C = \{2, 3, 5\}$$

Then:

$A \cup C = \{2, 3, 4, 5, 6\}$ is the event that an even or a prime number occurs;

$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs;

$C^c = \{1, 4, 6\}$ is the event that a prime number does not occur.

Note that A and B are mutually exclusive: $A \cap B = \emptyset$; in other words, an even number and an odd number cannot occur simultaneously.

2.

Experiment: Toss a coin 3 times and observe the sequence of heads (H) and tails (T) that appears. The sample space S consists of eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same:

$$A = \{HHH, HHT, THH\} \quad \text{and} \quad B = \{HHH, TTT\}$$

Then $A \cap B = \{HHH\}$ is the elementary event in which only heads appear. The event that 5 heads appear is the empty set $\emptyset$.

Experiment: Toss a coin 3 times and observe the sequence of heads (H) and tails (T) that appears. The sample space S consists of eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

2.2. Axioms, Interpretations, and Properties of Probability

1. **For any event A, $P(A) \geq 0$.**
2. **$P(S)=1$**
3. **If A and B are mutually exclusive events then**

$$P(A \cup B) = P(A) + P(B)$$

4. **If $A_1, A_2, \ldots\ldots$ is a sequence of mutually exclusive events**

then $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \cdots$

Theorem 1: If $\emptyset$ is the empty set, then $P(\emptyset) = 0$.

Proof: Let A be any set; then A and $\emptyset$ are disjoint and $A \cup \emptyset = A$.

$$P(A) = P(A \cup \emptyset) = P(A) + P(\emptyset)$$

Subtracting $P(A)$ from both sides gives our result.

Theorem 2: If A^c is the complement of an event A, then $P(A^c) = 1 - P(A)$.

Proof: The sample space S can be decomposed into the mutually exclusive events A and A^c; that is, $S = A \cup A^c$. By $[\mathbf{P}_2]$ and $[\mathbf{P}_3]$ we obtain

$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

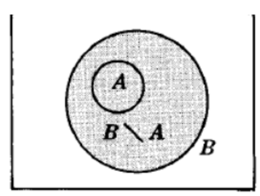
from which our result follows.

Theorem 3: If $A \subset B$, then $P(A) \leq P(B)$.

Proof. If $A \subset B$, then B can be decomposed into the mutually exclusive events A and $B \setminus A$ (as illustrated on the right). Thus

$$P(B) = P(A) + P(B \setminus A)$$

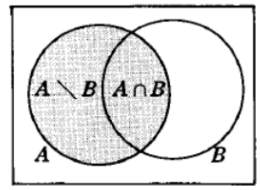
The result now follows from the fact that $P(B \setminus A) \geq 0$.



B is shaded.

Theorem 4: If A and B are any two events, then

$$P(A \setminus B) = P(A) - P(A \cap B)$$



A is shaded.

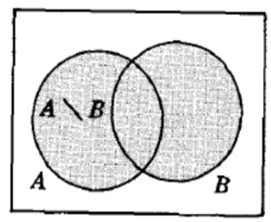
Proof. Now A can be decomposed into the mutually exclusive events $A \setminus B$ and $A \cap B$; that is, $A = (A \setminus B) \cup (A \cap B)$. Thus by [$\mathbf{P}_3$],

$$P(A) = P(A \setminus B) + P(A \cap B)$$

from which our result follows.

Theorem 5: If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



$A \cup B$ is shaded.

Proof. Note that $A \cup B$ can be decomposed into the mutually exclusive events $A \setminus B$ and B; that is, $A \cup B = (A \setminus B) \cup B$. Thus by [$\mathbf{P}_3$] and Theorem 3.4,

$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(B) \\ &= P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

which is the desired result.

Applying the above theorem twice (Problem 3.23) we obtain

Corollary 6: For any events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

2.3. Finite Probability Spaces

Let S be a finite sample space; say, $S = \{a_1, a_2, \ldots, a_n\}$. A finite probability space is obtained by assigning to each point $a_i \in S$ a real number p_i, called the *probability* of a_i, satisfying the following properties:

(i) each p_i is nonnegative, $p_i \geq 0$

(ii) the sum of the p_i is one, $p_1 + p_2 + \cdots + p_n = 1$.

The *probability* $P(A)$ of any event A, is then defined to be the sum of the probabilities of the points in A. For notational convenience we write $P(a_i)$ for $P(\{a_i\})$.

Example

1.

Let three coins be tossed and the number of heads observed; then the sample space is $S = \{0, 1, 2, 3\}$. We obtain a probability space by the following assignment

$$P(0) = \tfrac{1}{8}, \quad P(1) = \tfrac{3}{8}, \quad P(2) = \tfrac{3}{8} \quad \text{and} \quad P(3) = \tfrac{1}{8}$$

since each probability is nonnegative and the sum of the probabilities is 1. Let A be the event that at least one head appears and let B be the event that all heads or all tails appear:

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{0, 3\}$$

Then, by definition,

$$P(A) = P(1) + P(2) + P(3) = \tfrac{3}{8} + \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{7}{8}$$

and

$$P(B) = P(0) + P(3) = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}$$

2.

Three horses *A*, *B* and *C* are in a race ; *A* is twice as likely to win as *B* and *B* is twice as likely to win as*C* . What are their respective probabilities of winning i.e P(*A*),P(*B*) and P(*C*)

Let $P(C) = p$; since B is twice as likely to win as C, $P(B) = 2p$; and since A is twice as likely to win as B, $P(A) = 2P(B) = 2(2p) = 4p$. Now the sum of the probabilities must be 1; hence

$$p + 2p + 4p = 1 \quad \text{or} \quad 7p = 1 \quad \text{or} \quad p = \tfrac{1}{7}$$

Accordingly,

$$P(A) = 4p = \tfrac{4}{7}, \quad P(B) = 2p = \tfrac{2}{7}, \quad P(C) = p = \tfrac{1}{7}$$

Question: What is the probability that B or C wins, i.e. $P(\{B, C\})$? By definition

$$P(\{B, C\}) = P(B) + P(C) = \tfrac{2}{7} + \tfrac{1}{7} = \tfrac{3}{7}$$

3.

Let a die be weighted so that the probability of a number appearing when the die is tossed is proportional to the given number (e.g. 6 has twice the probability of appearing as 3). Let $A = \{\text{even number}\}$, $B = \{\text{prime number}\}$, $C = \{\text{odd number}\}$.

(i) Describe the probability space, i.e. find the probability of each sample point.

(ii) Find $P(A)$, $P(B)$ and $P(C)$.

(iii) Find the probability that: (*a*) an even or prime number occurs; (*b*) an odd prime number occurs; (*c*) A but not B occurs.

(i) Let $P(1) = p$. Then $P(2) = 2p$, $P(3) = 3p$, $P(4) = 4p$, $P(5) = 5p$ and $P(6) = 6p$. Since the sum of the probabilities must be one, we obtain $p + 2p + 3p + 4p + 5p + 6p = 1$ or $p = 1/21$. Thus

$$P(1) = \tfrac{1}{21}, \quad P(2) = \tfrac{2}{21}, \quad P(3) = \tfrac{1}{7}, \quad P(4) = \tfrac{4}{21}, \quad P(5) = \tfrac{5}{21}, \quad P(6) = \tfrac{2}{7}$$

(ii) $P(A) = P(\{2,4,6\}) = \frac{4}{7}$, $\quad P(B) = P(\{2,3,5\}) = \frac{10}{21}$, $\quad P(C) = P(\{1,3,5\}) = \frac{3}{7}$.

(iii) (*a*) The event that an even or prime number occurs is $A \cup B = \{2,4,6,3,5\}$, or that 1 does not occur. Thus $P(A \cup B) = 1 - P(1) = \frac{20}{21}$.

(*b*) The event that an odd prime number occurs is $B \cap C = \{3,5\}$. Thus $P(B \cap C) = P(\{3,5\}) = \frac{8}{21}$.

<Best Regards>