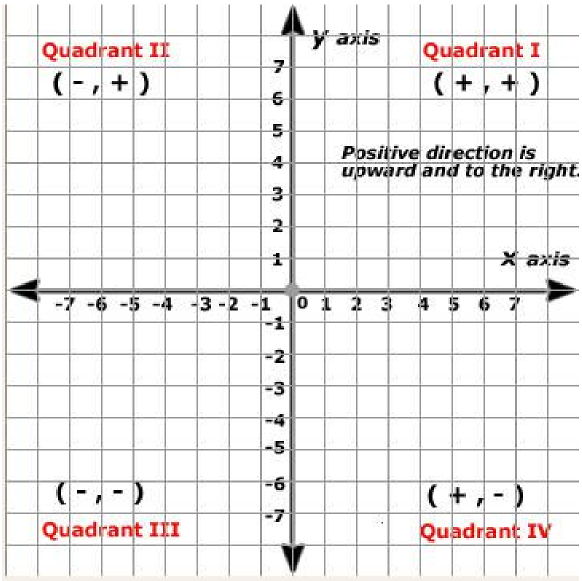
Cartesian Coordinates

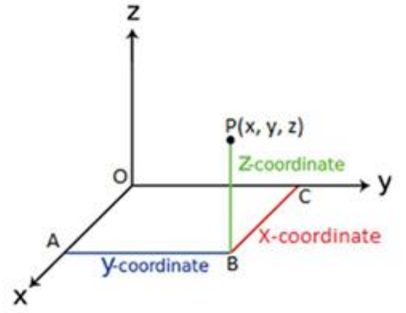
Cartesian Coordinates, also called rectangular coordinates, pinpoint a location in a plane or 3D space. Imagine a flat surface (2D plane) with two lines (axes) crossing at a point called the origin (0,0) - typically, the x-axis (horizontal) and y-axis (vertical). Each axis is like a number line stretching infinitely both ways, so any real number can be a coordinate.

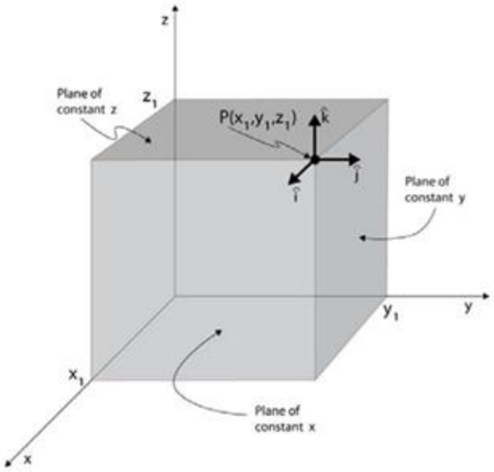
A 2D coordinate (x, y) tells you how far to move horizontally (x) and vertically (y) from the origin to hit the point.



1. Cartesian Coordinate System (Rectangular)

In rectangular coordinates a point P is specified by x, y, and z, where these values are all measured from the origin (see figure at right). A vector at the point P is specified in terms of three mutually perpendicular components with unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ (also called $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$). The unit vectors $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ form a righthanded set; that is, if you push $\hat{\boldsymbol{\imath}}$ into $\hat{\boldsymbol{\jmath}}$ with your right hand, your right thumb will point along $\hat{\boldsymbol{k}}$ direction.





- Position in 3D

$$(x, y, z)$$

- Position Vector

$$\vec{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}$$

- Magnitude / Distance

$$r = \sqrt{x^2 + y^2 + z^2}$$

- Unit Vector

$$\hat{\boldsymbol{r}} = \frac{\vec{\boldsymbol{r}}}{r} = \frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{\sqrt{x^2 + y^2 + z^2}}$$

- Gradient

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$$

- Divergence

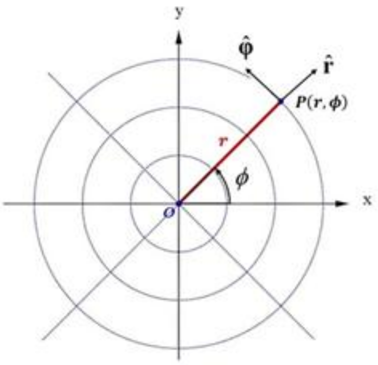
$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

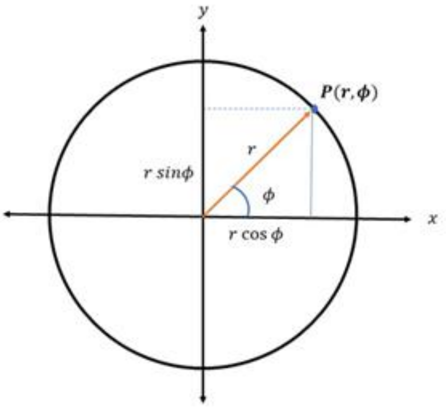
- Curl (Determinant Form)

$$\nabla \times \vec{\boldsymbol{A}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{\imath}} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\hat{\boldsymbol{\jmath}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{k}}$$

2. Polar Coordinate System

Polar coordinates are a two-dimensional (2D) coordinate system used to describe the position of a point in a plane in terms of its distance from the origin and its angular direction. A point is specified by (r, ϕ), where represents the radial distance and ϕ is the angle measured from the positive x-axis. This system is particularly useful for problems involving circular motion and radial symmetry.





- Position in Polar Coordinates: (r, ϕ)
- Relation to Cartesian Coordinates

$$x = r\cos\,\phi, \qquad y = r\sin\,\phi$$
$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

- Position Vector

$$\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}}$$

- Unit Vectors

$$\hat{\boldsymbol{r}} = \cos\,\phi\hat{\boldsymbol{\imath}} + \sin\,\phi\hat{\boldsymbol{\jmath}}$$
$$\hat{\boldsymbol{\phi}} = -\sin\,\phi\hat{\boldsymbol{\imath}} + \cos\,\phi\hat{\boldsymbol{\jmath}}$$

- Important Note:

$\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}$ depend on ϕ. The unit vector

$$\hat{\boldsymbol{\phi}} = \frac{1}{r}\frac{\partial\vec{\boldsymbol{r}}}{\partial\phi},$$

is defined as the normalized rate of change of the position vector with respect to the angular coordinate ϕ; it points in the direction of increasing ϕ (tangent to the circle of radius r).

Example1: A particle moves in a circular path in a plane. At a certain instant, its position in Cartesian coordinates is given by $(x, y) = (3,3\sqrt{3})$.

Find the radial distance r, the angle ϕ, the position vector, and the unit vectors.

Solution:

Radial distance : $r = \sqrt{x^2 + y^2} = \sqrt{9 + 27} = 6$

The angle: $\phi = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(\sqrt{3}) = 60°$

Position vector: $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} = 3\hat{\boldsymbol{\imath}} + 3\sqrt{3}\hat{\boldsymbol{\jmath}}$

Unit vectors: $\hat{\boldsymbol{r}} = \cos\,\phi\hat{\boldsymbol{\imath}} + \sin\,\phi\hat{\boldsymbol{\jmath}} = \frac{1}{2}\hat{\boldsymbol{\imath}} + \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}}$

$$\hat{\boldsymbol{\phi}} = -\sin\,\phi\hat{\boldsymbol{\imath}} + \cos\,\phi\hat{\boldsymbol{\jmath}} = -\frac{\sqrt{3}}{2}\hat{\boldsymbol{\imath}} + \frac{1}{2}\hat{\boldsymbol{\jmath}}$$

Example2: Derive the del operator in polar coordinates starting from its cartesian form.

Solution:

We start from the cartesian form of the del operator:

$$\nabla = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y}$$

We express the derivatives in terms of r and ϕ using the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} \qquad (1)$$

The radial coordinate is defined as:

$$r = \sqrt{x^2 + y^2}$$

Thus,

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)^{1/2} = \frac{1}{2}(x^2 + y^2)^{-1/2} \cdot 2x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x}{r} = \cos\,\phi \qquad (2)$$

For the angular coordinate:

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$

We differentiate:

$$\frac{\partial \phi}{\partial x} = \frac{1}{1+(y/x)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right) = \frac{x^2}{x^2+y^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{y}{r^2} = -\frac{\sin\ \phi}{r} \quad (3)$$

Using all the above equation with Eq. (1)

Therefore,

$$\frac{\partial}{\partial x} = \cos\ \phi \frac{\partial}{\partial r} - \frac{\sin\ \phi}{r}\frac{\partial}{\partial \phi} \quad (4)$$

Similarly:

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\ \phi$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x} = \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2} = \frac{x}{r^2} = \frac{\cos\ \phi}{r}$$

Thus,

$$\frac{\partial}{\partial y} = \sin\ \phi \frac{\partial}{\partial r} + \frac{\cos\ \phi}{r}\frac{\partial}{\partial \phi} \qquad (5)$$

Combining both parts from (4) and (5) in the cartesian form:

$$\nabla = (\cos\ \phi \hat{\boldsymbol{\imath}} + \sin\ \phi \hat{\boldsymbol{\jmath}})\frac{\partial}{\partial r} + (-\sin\ \phi \hat{\boldsymbol{\imath}} + \cos\ \phi \hat{\boldsymbol{\jmath}})\frac{1}{r}\frac{\partial}{\partial \phi}$$

Recognizing:

$$\hat{\boldsymbol{r}} = \cos\ \phi \hat{\boldsymbol{\imath}} + \sin\ \phi \hat{\boldsymbol{\jmath}}, \qquad \hat{\boldsymbol{\phi}} = -\sin\ \phi \hat{\boldsymbol{\imath}} + \cos\ \phi \hat{\boldsymbol{\jmath}}$$

We obtain:

$$\nabla = \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial \phi}$$

- Gradient for a scalar function is

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}$$

Suppose we have a vector field in polar coordinates defined as:

$$\vec{\boldsymbol{A}} = A_r\hat{\boldsymbol{r}} + A_\phi\hat{\boldsymbol{\phi}}$$

- Divergence

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi}$$

- Curl

$$\nabla \times \vec{\boldsymbol{A}} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]\hat{\boldsymbol{k}}$$

Example 3: A vector field is given in polar coordinates as: $\vec{\boldsymbol{A}} = r^2\hat{\boldsymbol{r}} + r\sin\phi\,\hat{\boldsymbol{\phi}}$.

Find the divergence of the vector field.

Solution:

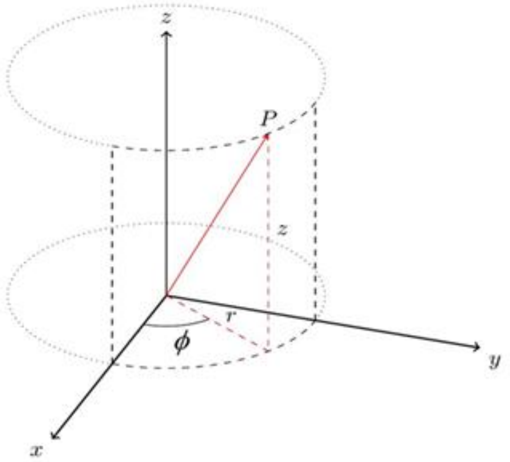
Divergence: $\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi}$

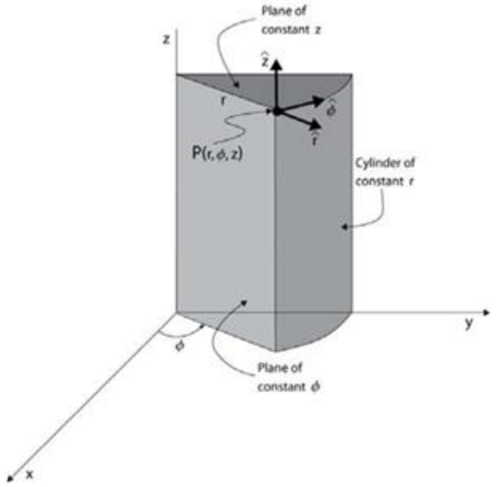
$$= \frac{1}{r}\frac{\partial}{\partial r}(r^3) + \frac{1}{r}\frac{\partial(r\sin\phi)}{\partial \phi}$$

$$= \frac{1}{r}(3r^2) + \frac{1}{r}(r\cos\phi) = 3r + \cos\phi$$

3. Cylindrical Coordinate System

Cylindrical coordinates are a three-dimensional (3D) coordinate system that extend polar coordinates by adding a vertical component. A point is described by (r, ϕ, z), where r and ϕ define the position in the plane using polar coordinates, and z represents the height along the vertical axis.





- A point in cylindrical coordinates is written as: (r, ϕ, z)
- The relation to Cartesian coordinates is:

$$x = r\cos\,\phi, y = r\sin\,\phi, z = z$$
$$r = \sqrt{x^2 + y^2}, \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

- The position vector is: $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}} + z\hat{\boldsymbol{z}}$
- The unit vectors are:

$$\hat{\boldsymbol{r}} = \cos\,\phi\hat{\boldsymbol{\imath}} + \sin\,\phi\hat{\boldsymbol{\jmath}}$$
$$\hat{\phi} = -\sin\,\phi\hat{\boldsymbol{\imath}} + \cos\,\phi\hat{\boldsymbol{\jmath}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{k}}$$

- The gradient for scalar function is:

$$\nabla f = \frac{\partial f}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$$

Suppose we have a vector field in cylindrical coordinates defined as:

$$\vec{\boldsymbol{A}} = A_r\hat{\boldsymbol{r}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\boldsymbol{z}}$$

- The divergence is:

$$\nabla \cdot \vec{\mathbf{A}} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

- The curl is:

$$\nabla \times \vec{\mathbf{A}} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{\boldsymbol{z}}$$

Example 4: A vector field is given in cylindrical coordinates as: $\vec{\boldsymbol{A}} = r\hat{\boldsymbol{r}} + r\phi\hat{\boldsymbol{\phi}} + z\hat{\boldsymbol{z}}$

Find the divergence and curl of the vector field.

Solution: We identify:

$$A_r = r, A_\phi = r\phi, A_z = z$$

The divergence in cylindrical coordinates is:

$$\nabla \cdot \vec{\boldsymbol{A}} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Substituting:

$$\begin{aligned}\nabla \cdot \vec{\boldsymbol{A}} &= \frac{1}{r}\frac{\partial}{\partial r}(r\,r) + \frac{1}{r}\frac{\partial}{\partial \phi}(r\phi) + \frac{\partial}{\partial z}(z) \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r^2) + \frac{1}{r}(r) + 1 \\ &= \frac{1}{r}(2r) + 1 + 1 = 4\end{aligned}$$

The curl in cylindrical coordinates is:

$$\nabla \times \vec{\boldsymbol{A}} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{r}} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\boldsymbol{\phi}} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right)\hat{\boldsymbol{z}}$$

Substituting:

$$\frac{\partial A_z}{\partial \phi} = 0, \qquad \frac{\partial A_\phi}{\partial z} = 0, \qquad \frac{\partial A_r}{\partial z} = 0, \qquad \frac{\partial A_z}{\partial r} = 0, \qquad \frac{\partial A_r}{\partial \phi} = 0$$

$$\frac{\partial}{\partial r}\left(rA_\phi\right) = \frac{\partial}{\partial r}(r^2\phi) = 2r\phi$$

$$\frac{\partial A_r}{\partial \phi} = 0$$

$$\nabla \times \vec{\boldsymbol{A}} = 2\phi\hat{\mathbf{z}}$$

H. W. Calculate the following:

1. Find the cartesian coordinates (x, y) and the position vector of the point given in polar coordinates as

$$(r, \phi) = \left(2, \frac{\pi}{3}\right)$$

2. Find the cartesian coordinates (x, y, z) and the position vector of the point given in cylindrical coordinates as

$$(r, \phi, z) = \left(2, \frac{\pi}{3}, 4\right)$$

3. A vector field is given in cylindrical coordinates as $\vec{\boldsymbol{A}} = r^2\hat{\mathbf{r}} + r\sin\,\phi\hat{\boldsymbol{\phi}} + z^2\hat{\mathbf{z}}$.

   Find the divergence and curl.