

## Derivatives

### Rules for finding derivatives

#### 1. Constant Rule

The derivative of a constant is always zero. That is if  $f(x) = c$  then  $f'(x) = 0$ .

**2. Power Rule** The derivative of a power function,  $f(x) = x^n$ . Here  $n$  is a number of any kind: integer, rational, positive, negative, even irrational, if  $f(x) = x^n$ , then the derivative is  $f'(x) = nx^{n-1}$

Examples:

No	$y$	$\frac{dy}{dx}$
1.	$y = x^4$	$\frac{dy}{dx} = 4x^3$
2.	$y = x^{-4}$	$\frac{dy}{dx} = -4x^{-5} = -\frac{4}{x^5}$
3.	$y = \frac{1}{x^2} = x^{-2}$	$\frac{dy}{dx} = -2x^{-3} = -\frac{2}{x^3}$
4.	$y = x^{\frac{3}{5}}$	$\frac{dy}{dx} = \frac{3}{5}x^{\frac{3}{5}-1} = \frac{3}{5}x^{-\frac{2}{5}}$
5.	$y = 3x^{-\frac{5}{3}}$	$\frac{dy}{dx} = -5x^{-\frac{8}{3}}$

**3. Multiplication by constant:** If  $g(x) = cf(x)$  then  $g'(x) = cf'(x)$ .

**4. Sum Rule:** If  $h(x) = f(x) + g(x)$ , then  $h'(x) = f'(x) + g'(x)$ .

**5. Difference Rule:** If  $h(x) = f(x) - g(x)$ , then  $h'(x) = f'(x) - g'(x)$ .

**6. Product Rule:** The derivative of the product of two functions is not the product of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx}(f(x) \times g(x)) = f(x) \times g'(x) + f'(x) \times g(x)$$

**Example 6:** Find derivative of the function  $y = (3x^2 + 5)(2x^3 - 5x - 4)$

$$\frac{dy}{dx} = (3x^2 + 5) \times (6x^2 - 5) + (2x^3 - 5x - 4) \times 6x$$

**7. Quotient Rule:** The derivative of the quotient of two functions is not the quotient of the functions' derivatives; rather, it is described by the equation below:

$$\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g(x) \times f'(x) - f(x) \times g'(x)}{(g(x))^2}$$

**Example 7:** Find derivative of the function

$$y = \frac{x^2 - 1}{x^2 + 1}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{(x^2 + 1) \times 2x - (x^2 - 1) \times 2x}{(x^2 + 1)^2} = \frac{2x^3 + 2x - (2x^3 - 2x)}{(x^2 + 1)^2} \\ &= \frac{2x^3 + 2x - 2x^3 + 2x}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2} \end{aligned}$$

### 8. Chain Rule:

Suppose that we have two functions  $f(x)$  and  $g(x)$  and they are both differentiable.

1.If we define  $F(x) = (f \circ g)(x)$  then the derivative of  $F(x)$  is,

$$F'(x) = f'(g(x))g'(x)$$

2.If we have  $y = f(u)$  and  $u = g(x)$  then the derivative of  $y$  is,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

**Example 8:** Find derivatives of the functions

$$1. y = (2x + 3)^4 \Rightarrow \frac{dy}{dx} = 4(2x + 3)^3 \times 2 = 8(2x + 3)^3$$

$$2. y = \sqrt{x^2 + 3x} \Rightarrow y = (x^2 + 3x)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2} (x^2 + 3x)^{-1/2} (2x + 3) = \frac{(2x + 3)}{2\sqrt{x^2 + 3x}}$$

$$3. y = \frac{x}{\sqrt{x^2 + 1}} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{x^2 + 1} \times 1 - x \times (1/2)(x^2 + 1)^{-1/2} \times 2x}{x^2 + 1}$$

$$= \frac{\sqrt{x^2 + 1} - \frac{x^2}{\sqrt{x^2 + 1}}}{x^2 + 1} = \frac{\frac{x^2 + 1 - x^2}{\sqrt{x^2 + 1}}}{x^2 + 1} = \frac{1}{\sqrt{x^2 + 1}}$$

**Example 9:** If  $y = u^2 - 2u$  and  $u = \sqrt{3x + 1}$ , find  $\frac{dy}{dx}$

$$\frac{dy}{du} = 2u - 2 = 2(u - 1) \quad \text{and} \quad \frac{du}{dx} = \frac{3}{2\sqrt{3x + 1}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 2(u - 1) \times \frac{3}{2\sqrt{3x + 1}} = \frac{3(u - 1)}{\sqrt{3x + 1}} = \frac{3(\sqrt{3x + 1} - 1)}{\sqrt{3x + 1}}$$

**Example 10:** If  $y = t + \frac{1}{t}$  and  $x = t - \frac{1}{t}$ , find  $\frac{dy}{dx}$

$$\frac{dy}{dt} = 1 - \frac{1}{t^2} = \frac{t^2 - 1}{t^2} \quad \text{and} \quad \frac{dx}{dt} = 1 + \frac{1}{t^2} = \frac{t^2 + 1}{t^2}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{t^2 - 1}{t^2} \times \frac{t^2}{t^2 + 1} = \frac{t^2 - 1}{t^2 + 1}$$

### Higher Derivatives

If the derivative  $f'(x)$  of a function  $f(x)$  exists in the domain of  $f(x)$ , then we have a new function. Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the **second derivative**. It is denoted  $f''(x)$ . The derivative of the second derivative is called the third derivative, written  $f'''(x)$ , and so on.

The  $n$ th derivative of  $f(x)$  is denoted  $f^{(n)}(x)$ . Thus

Leibniz' notation for the  $n$ th derivative of  $y = f(x)$  is  $\frac{d^n y}{dx^n}$ .

Be careful to distinguish the second derivative from the square of the first derivative. Usually

$$\frac{d^2 y}{dx^2} \neq \left( \frac{dy}{dx} \right)^2$$

**Example 11:** Find  $f'(x)$ ,  $f''(x)$ ,  $f^{(3)}(x)$  and  $f^{(4)}(x)$  for  $f(x) = 2x^3 + 3x^2 - 4x + 5$

$$f'(x) = 6x^2 + 6x - 4$$

$$f''(x) = 12x + 6$$

$$f^{(3)}(x) = 12$$

$$f^{(4)}(x) = 0$$

**Example 12:** Compute the first, second and third derivatives of  $y = \sqrt{x+2}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4(x+2)^{3/2}}$$

$$\frac{d^3y}{dx^3} = \frac{3}{8(x+2)^{5/2}}$$

### Exercises

Find derivative in each of the following problems (1 – 4)

1.  $y = (x^2 - 1)^4$

2.  $y = x^2\sqrt{2x^2 + 3}$

3.  $y = \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$

4.  $y = \sqrt{\frac{1-x}{x^2+1}}$

Compute the first, second and third derivatives in the following problems (5 – 10)

5.  $y = 2x^3 - 5x^2 + 3x$

8.  $y = \sqrt{2x+3}$

9.  $y = x\sqrt{x}$

10.  $y = (x^2 + 2)^{5/2}$

11. If  $y = u\sqrt{2u+5}$  and  $x = (4u)^{1/3}$  find  $\frac{dy}{dx}$

12. If  $u = s + \sqrt{s}$  and  $v = s - \sqrt{s}$  find  $\frac{du}{dv}$