Limits of Functions

A limit is the value that a function or sequence "approaches" as the input or index approaches some value. We say that the limit of $f(x)$ is L as x approaches a and write this as $\lim_{x\to a} f(x) = L$.

Properties of the limit

If a and c are constants, $\lim_{x\to a} f_1(x) = L_1$ and $\lim_{x\to a} f_2(x) = L_2$, then

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a}\big(f_1(x) + f_2(x)\big) = L_1 + L_2$
4. $\lim_{x\to a}\big(f_1(x) - f_2(x)\big) = L_1 - L_2$
5. $\lim_{x\to a}\big(f_1(x).f_2(x)\big) = L_1.L_2$
6. $\lim_{x\to a}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{L_1}{L_2}$; $L_2 \neq 0$
7. $\lim_{x\to a} f_1\big(f_2(x)\big) = f_1(L_2)$
8. $\lim_{x\to a}\big(f_1(x)\big)^n = {L_1}^n$

Examples: Find the following limit

1. $\lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{x-2} = \lim_{x\to 2}\frac{\sqrt{x}-\sqrt{2}}{(\sqrt{x}-\sqrt{2})(\sqrt{x}+\sqrt{2})} = \lim_{x\to 2}\frac{1}{(\sqrt{x}+\sqrt{2})} = \frac{1}{2\sqrt{2}}$

2. $\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5}$

Divide top and bottom by x^2, then we get

$$\lim_{x\to\infty}\frac{5x^2+3}{7x^2+2x-5} = \lim_{x\to\infty}\frac{5+(3/x^2)}{7+(2/x)-(5/x^2)} = \frac{5}{7}$$

3. $\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} = \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\times\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}$

$$= \lim_{x\to 0}\frac{x^2+4-4}{x\left(\sqrt{x^2+4}+2\right)} = \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} = \lim_{x\to 0}\frac{1}{\left(\sqrt{x^2+4}+2\right)} = \frac{1}{4}$$

4. $\lim_{x\to\infty}\frac{3x}{\sqrt{4x^2+1}} = \lim_{x\to\infty}\frac{\frac{3x}{x}}{\frac{\sqrt{4x^2+1}}{x}} = \lim_{x\to\infty}\frac{3}{\sqrt{\frac{4x^2+1}{x^2}}} = \lim_{x\to\infty}\frac{3}{\sqrt{4+\frac{1}{x^2}}} = \frac{3}{2}$

$$5.\ \lim_{x\to\infty}\frac{(2x+1)^4}{(3x^2+5x-1)^2}=\left(\lim_{x\to\infty}\frac{(2x+1)^2}{3x^2+5x-1}\right)^2=\left(\lim_{x\to\infty}\frac{4x^2+4x+1}{3x^2+5x-1}\right)^2$$

$$=\left(\lim_{x\to\infty}\frac{4+(4/x)+(1/x^2)}{3+(5/x)-(1/x^2)}\right)^2=\left(\frac{4}{3}\right)^2=\frac{16}{9}$$

Exercises

Evaluate the limits

1. $\lim_{x\to 2}\frac{2-\sqrt{3x-2}}{\sqrt{2x+5}-3}$

2. $\lim_{x\to\infty}\frac{3x^3-2x+1}{x^3+3x^2+x}$

3. $\lim_{x\to\infty}\frac{(x+4)^6}{(4x^2+4x+1)^3}$

4. $\lim_{x\to 1}\frac{\sqrt{x^2+15}-4}{x-1}$