Continuous Random Variables

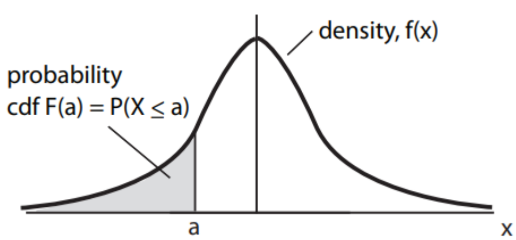
Definitions

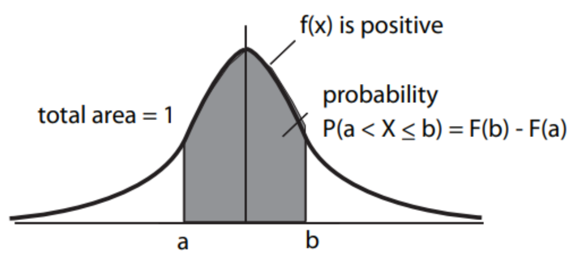
Random variable X is *continuous* if *probability density function* (pdf) f is continuous at all but a finite number of points and possesses the following properties:

- $f(x) \geq 0$, for all x,
- $\int_{-\infty}^{\infty} f(x)\, dx = 1$,
- $P(a < X \leq b) = \int_a^b f(x)\, dx$

The *(cumulative) distribution function* (cdf) for random variable X is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt,$$





The *expected value* or *mean* of random variable X is given by

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx,$$

the *variance* is

$$\sigma^2 = Var(X) = E[(X-\mu)^2] = E(X^2) - [E(X)]^2 = E(X^2) - \mu^2$$

Problem

Let X be a random variable with PDF given by

$$f_X(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the constant c.
b. Find EX and Var(X).
c. Find $P(X \geq \frac{1}{2})$.

Solution

a. To find c, we can use $\int_{-\infty}^{\infty} f_X(u)du = 1$:

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} f_X(u)du \\ &= \int_{-1}^{1} cu^2 du \\ &= \frac{2}{3}c. \end{aligned}$$

Thus, we must have $c = \frac{3}{2}$.

b. To find EX, we can write

$$\begin{aligned} EX &= \int_{-1}^{1} u f_X(u)du \\ &= \frac{3}{2}\int_{-1}^{1} u^3 du \\ &= 0. \end{aligned}$$

In fact, we could have guessed $EX = 0$ because the PDF is symmetric around $x = 0$. To find Var(X), we have

$$\begin{aligned} \text{Var}(X) &= EX^2 - (EX)^2 = EX^2 \\ &= \int_{-1}^{1} u^2 f_X(u)du \\ &= \frac{3}{2}\int_{-1}^{1} u^4 du \\ &= \frac{3}{5}. \end{aligned}$$

c. To find $P(X \geq \frac{1}{2})$, we can write

$$P(X \geq \frac{1}{2}) = \frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx = \frac{7}{16}.$$

Problem

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(X \leq \frac{2}{3} | X > \frac{1}{3})$.

Solution

We have

$$\begin{aligned} P(X \leq \tfrac{2}{3} | X > \tfrac{1}{3}) &= \frac{P(\frac{1}{3} < X \leq \frac{2}{3})}{P(X > \frac{1}{3})} \\ &= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 dx}{\int_{\frac{1}{3}}^{1} 4x^3 dx} \\ &= \frac{3}{16}. \end{aligned}$$

Problem

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 \left(2x + \frac{3}{2}\right) & 0 < x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

If $Y = \frac{2}{X} + 3$, find Var(Y).

Solution

First, note that

$$\text{Var}(Y) = \text{Var}\left(\frac{2}{X} + 3\right) = 4\text{Var}\left(\frac{1}{X}\right), \qquad \text{using Equation}$$

Thus, it suffices to find Var$(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2$. Using LOTUS, we have

$$E\left[\frac{1}{X}\right] = \int_0^1 x\left(2x + \frac{3}{2}\right)dx = \frac{17}{12}$$

$$E\left[\frac{1}{X^2}\right] = \int_0^1 \left(2x + \frac{3}{2}\right)dx = \frac{5}{2}.$$

Thus, $\mathrm{Var}\left(\frac{1}{X}\right) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2 = \frac{71}{144}$. So, we obtain

$$\mathrm{Var}(Y) = 4\mathrm{Var}\left(\frac{1}{X}\right) = \frac{71}{36}.$$

The Uniform Distributions

Two special probability density functions are discussed: uniform and exponential. The continuous *uniform* (rectangular) distribution of random variable X has density

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{elsewhere,} \end{cases}$$

where "X is $U(a, b)$" means "X is uniform over [a,b]", distribution function,

$$F(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \le x \le b, \\ 1 & x > b, \end{cases}$$

and its expected value (mean), variance and standard deviation are,

$$\mu = E(X) = \frac{a+b}{2}, \quad \sigma^2 = Var(X) = \frac{(b-a)^2}{12}, \quad \sigma = \sqrt{Var(X)},$$

Problem:

A bus arrives at a bus stop every 20 minutes. A passenger arrives at the bus stop at a random time. Let X be the number of minutes the passenger has to wait for the bus.

Assume $X \sim \text{Uniform}(0, 20)$.

Questions:

1. What is the probability that the passenger waits more than 10 minutes?
2. What is the expected (average) waiting time?
3. What is the probability the passenger waits between 5 and 15 minutes?

PDF (Probability Density Function):

$$f(x) = \frac{1}{b-a} = \frac{1}{20-0} = \frac{1}{20}, \quad \text{for } 0 \leq x \leq 20$$

. Expected waiting time:

$$E[X] = \frac{a+b}{2} = \frac{0+20}{2} = 10$$

. Probability the passenger waits between 5 and 15 minutes:

$$P(5 \leq X \leq 15) = \int_{5}^{15} \frac{1}{20} dx = \frac{1}{20}(15-5) = \frac{10}{20} = 0.5$$