4. Limits

$\lim_{x\to\infty} f(x) = L_1$ *and* $\lim_{x\to\infty} g(x) = L_2$

where: L_1 and L_2 are real numbers, then:

(1)	$\lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = L_1 + L_2$	(7)	$\lim_{x\to 0} \cos x = 1$
(2)	$\lim_{x\to\infty} f(x) - \lim_{x\to\infty} g(x) = L_1 - L_2$	(8)	$\lim_{x\to 0} \frac{\sin x}{x} = 1$
(3)	$\lim_{x\to\infty} f(x) \times \lim_{x\to\infty} g(x) = L_1 \times L_2$	(9)	$\lim_{x\to 0} \tan x = 0$
(4)	$\lim_{x\to\infty} k f(x) = k L_1$ (k is any number)	(10)	$\lim_{x\to 0} \frac{\tan x}{x} = 1$
(5)	$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ if $L_2 \neq 0$	(11)	$\lim_{x\to\infty} \frac{\text{number}}{x} = 0$
(6)	$\lim_{x\to 0} \sin x = 0$	(12)	$\lim_{x\to 0} \frac{\text{number}}{x} = \infty$

Sandwich theorem: $f(x) \le g(x) \le h(x)$

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x) \le \lim_{x\to c} h(x)$$

$$L \le \lim_{x\to c} g(x) \le L \quad , \text{ then } \quad \lim_{x\to c} g(x) = L$$

$\frac{\infty}{\infty}$	$\frac{0}{0}$	$\frac{0}{\infty}$	$\frac{\infty}{0}$	$0 - 0$	$\infty - \infty$	$(\infty)^\infty$	$(\infty)^0$	$(0)^\infty$	$(0)^0$

Example 1: *Find* $\lim_{x\to\infty} \frac{3x^4+3x^2+2x}{x^4-x+1}$

Solution: $\lim_{x\to\infty} \frac{3x^4+3x^2+2x}{x^4-x+1} = \frac{\infty}{\infty}$,,,,,, then divided by x^4

$$\lim_{x \to \infty} \frac{3+\frac{3}{x^2}+\frac{2}{x^3}}{1-\frac{1}{x^3}+\frac{1}{x^4}} = \frac{3+0+0}{1-0+0} = 3$$

OR :::: by derivative

$$\lim_{x \to \infty} \frac{12x^3+6x-2}{4x^3-1} = \frac{\infty}{\infty} \;,,,,\; \lim_{x \to \infty} \frac{36x^2+6}{12x^2} = \frac{\infty}{\infty} \;,,,\; \lim_{x \to \infty} \frac{72x}{24x} = 3$$

Example 2: Find $\lim_{x \to \infty} \frac{x-\sin x}{x+1} = \frac{\infty}{\infty}$ (*not OK*)

Solution: $\lim_{x \to \infty} \frac{1-\frac{\sin x}{x}}{1+\frac{1}{x}} = \frac{1-0}{1+0} = 1$

Example 3: Find $\lim_{n \to \infty} \sqrt{n^2-n} - n$

Solution: $\lim_{n \to \infty} \sqrt{n^2-n} - n = \infty - \infty$ (*not OK*)

$$\lim_{n \to \infty} \left(\sqrt{n^2-n} - n\right) \frac{\sqrt{n^2-n}+n}{\sqrt{n^2-n}+n}$$

$$= \lim_{n \to \infty} \frac{n^2-n-n^2}{\sqrt{n^2-n}+n} = \lim_{n \to \infty} \frac{-n}{\sqrt{n^2-n}+n} = \frac{\infty}{\infty}$$

$$= \lim_{n \to \infty} \frac{-n \times \frac{1}{n}}{\frac{1}{n}\left(\sqrt{n^2-n}+n\right)} = \lim_{n \to \infty} \frac{-1}{\sqrt{1-\frac{1}{n}}+1} = \frac{-1}{\sqrt{1-0}+1} = -\frac{1}{2}$$

Example 4: Find $\lim_{x \to 0} \frac{\sin^3 3x}{x \sin^2 2x} = \frac{0}{0}$ (*not OK*)

Solution: $\lim_{x \to 0} \frac{1}{x} \times \frac{\sin^3 (3x)}{(3x)^3} \times \frac{(3x)^3}{1} \times \frac{(2x)^2}{\sin^2 (2x)} \times \frac{1}{(2x)^2}$

$$= \lim_{x \to 0} \frac{(3x)^3 \times 1}{x\,(2x)^2} = = \lim_{x \to 0} \frac{27x^3}{4x^3} = \frac{27}{4}$$

5. Trigonometric Functions:

(1)	$\sin^2 x + \cos^2 x = 1$
(2)	$\tan^2 x + 1 = \sec^2 x$
(3)	$\cot^2 x + 1 = \csc^2 x$
(4)	$\sin(x+y) = \sin x \cos y + \sin y \cos x$
(5)	$\sin(x-y) = \sin x \cos y - \sin y \cos x$
(6)	$\cos(x+y) = \cos x \cos y - \sin x \sin y$
(7)	$\cos(x-y) = \cos x \cos y + \sin x \sin y$
(8)	$\sin 2x = 2 \sin x \cos x$
(9)	$\sin(-x) = -\sin x \ , \cos(-x) = \cos x$
(10)	$\cos 2x = \cos^2 x - \sin^2 x$
(11)	$\cos^2 x = \frac{1+\cos 2x}{2}$
(12)	$\sin^2 x = \frac{1-\cos 2x}{2}$
(13)	$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
(14)	$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Example 4: Find $\lim_{x\to 0} \frac{\sqrt{\cos x}-1}{\sin^2 x} = \frac{0}{0}$ *(not OK)*

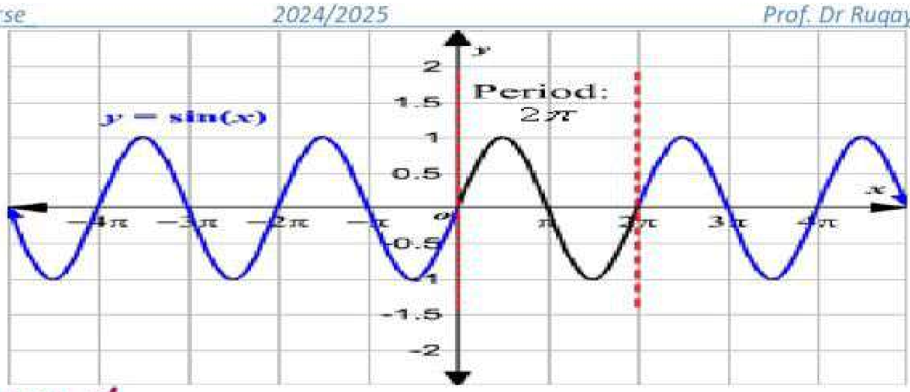
Solution:

$$\lim_{x\to 0} \frac{\sqrt{\cos x}-1}{\sin^2 x} \times \frac{\sqrt{\cos x}+1}{\sqrt{\cos x}+1} = \lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x\left(\sqrt{\cos x}+1\right)} = \frac{0}{0}$$

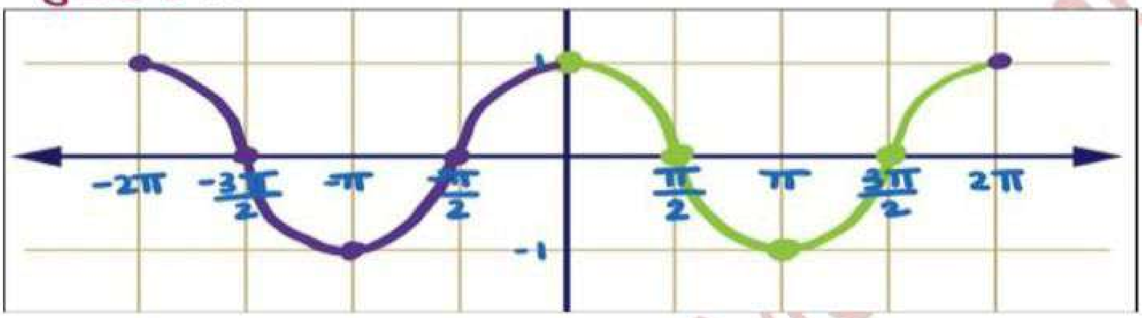
$$= \lim_{x\to 0} \frac{\cos x - 1}{\sin^2 x\left(\sqrt{\cos x}+1\right)} \times \frac{\cos x + 1}{\cos x + 1}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{\sin^2 x\left(\sqrt{\cos x}+1\right) \times (\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-\sin^2 x - 1}{\sin^2 x\left(\sqrt{\cos x}+1\right) \times (\cos x + 1)} = \frac{-1}{(1+1) \times (1+1)} = -\frac{1}{4}$$

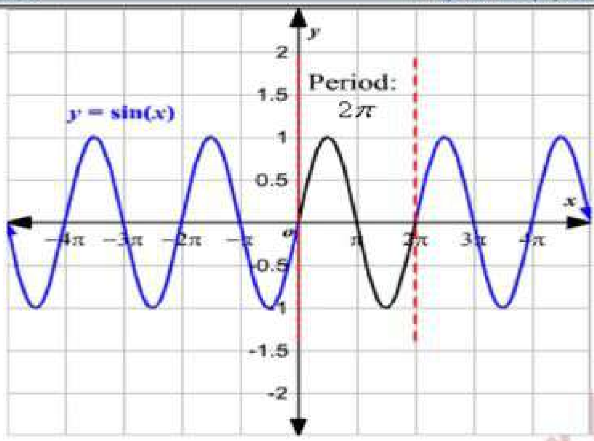


y = cos x

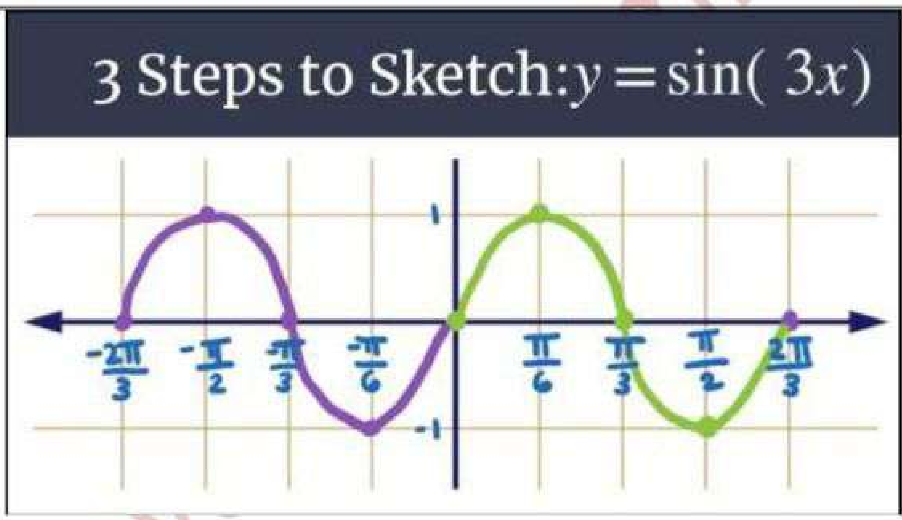


$(\pi\ ,2\ \pi)$	$\left(\frac{\pi}{2}\ ,\frac{3\ \pi}{2}\right)$
$\sin(\pi - x) = \sin x$	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$
$\cos(\pi - x) = -\cos x$	$\cos\left(\frac{\pi}{2} - x\right) = \sin x$
$\sin(\pi + x) = -\sin x$	$\sin\left(\frac{\pi}{2} + x\right) = \cos x$
$\cos(\pi + x) = -\cos x$	$\cos\left(\frac{\pi}{2} + x\right) = -\sin x$
$\sin(2\pi - x) = -\sin x$	$\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$
$\cos(2\pi - x) = \cos x$	$\cos\left(\frac{3\pi}{2} - x\right) = -\sin x$
$\sin(2\pi + x) = \sin x$	$\sin\left(\frac{3\pi}{2} + x\right) = -\cos x$
$\cos(2\pi + x) = \cos x$	$\cos\left(\frac{3\pi}{2} + x\right) = \sin x$

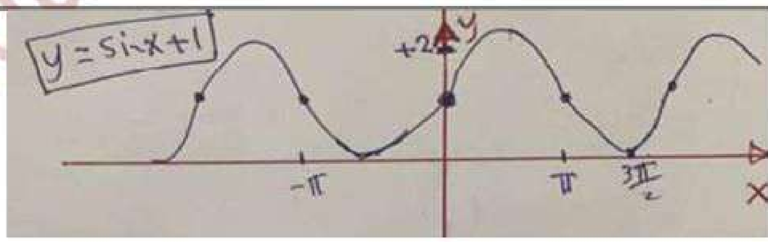
	x	$y = \sin x$
0	0	0
90	$\frac{\pi}{2}$	1
180	π	0
270	$\frac{3\pi}{2}$	- 1
360	2π	0



	x	$y = \sin 3x$
	0	0
	$\frac{\pi}{6}$	1
	$\frac{\pi}{3}$	0
	$\frac{3\pi}{6}$	- 1
	$\frac{2\pi}{3}$	0



	x	$y = \sin x + 1$
	0	1
	$\frac{\pi}{2}$	2
	π	1
	$\frac{3\pi}{2}$	0
	2π	1



Example 5: Find $\lim_{x \to \frac{\pi}{2}} (\sec x - \tan x)$

Solution: $\lim_{x \to \frac{\pi}{2}} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \left(\frac{1}{0} - \frac{1}{0}\right) = \infty - \infty \quad (not\ OK)$

$$= \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin x}{\cos x}\right) \times \frac{1+\sin x}{1+\sin x} = \lim_{x\to\frac{\pi}{2}} \left(\frac{1-\sin^2 x}{\cos x\ (1+\sin x)}\right)$$

Example 6: Find $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x-\frac{\pi}{4}} = \frac{0}{0}$

Solution: assume $y = x - \frac{\pi}{4}$, $x = y + \frac{\pi}{4}$, $\therefore x = \frac{\pi}{4}$

$$= \lim_{y\to 0} \frac{\sin\left(\frac{\pi}{4}+y\right) - \cos\left(\frac{\pi}{4}+y\right)}{y}$$

$$= \lim_{y\to 0} \frac{\left(\sin\frac{\pi}{4}\cos y + \sin y \cos\frac{\pi}{4}\right) - \left(\cos\frac{\pi}{4}\cos y - \sin\frac{\pi}{4}\sin y\right)}{y}$$

$$= \lim_{y\to 0} \frac{\frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y - \frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y}{y}$$

$$= \lim_{y\to 0} \frac{\frac{2}{\sqrt{2}}\sin y}{y} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Example 7: Find $\lim_{x\to 2} \frac{x-2}{\sin(x^2-4)} = \frac{0}{0}$

Solution: $\lim_{x\to 2} \frac{(x-2)\ (x+2)}{\sin(x^2-4)\ (x+2)} = \lim_{x\to 2} \frac{(x^2-4)}{\sin(x^2-4)\ (x+2)} = 1$

$$= \frac{1}{2+2} = \frac{1}{4}$$

Example 8: Find $\lim_{x\to \pi} \frac{1+\cos x}{(\pi-x)^2} = \frac{0}{0}$

Solution: $y = \pi - x$, $x = \pi - y$, $\therefore x = \pi \rightarrow y = 0$

$$= \lim_{y\to 0} \frac{1+\cos(\pi-y)}{y^2} = \lim_{y\to 0} \frac{1-\cos y}{y^2} = \lim_{y\to 0} \frac{2\sin^2\frac{y}{2}}{y^2}$$

$$= \lim_{y \to 0} 2 \times \frac{1}{4} \times \frac{\sin^2 \frac{y}{2}}{\left(\frac{y}{2}\right)^2} = 2 \times \frac{1}{4} \times 1 = \frac{1}{2} \text{ ,,,,,, } \quad \frac{1 - \cos y}{2} = \sin^2 \frac{y}{2}$$

Example 9: Find $\lim_{x \to 0} \frac{\tan x - \sin x}{x^3} = \frac{0}{0}$

Solution: $\lim_{x \to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} = \lim_{x \to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{x^3}$

$$= \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}\,(1 - \cos x)}{x^3} \times \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x\ (1 - \cos^2 x)}{\cos x\,(x^3)(1 + \cos x)} = \lim_{x \to 0} \frac{\sin x\ (\sin^2 x)}{\cos x\,(x^3)(1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^3 x}{\cos x\,(x^3)(1 + \cos x)} = \frac{1}{1\,(1 + 1)} = \frac{1}{2}$$