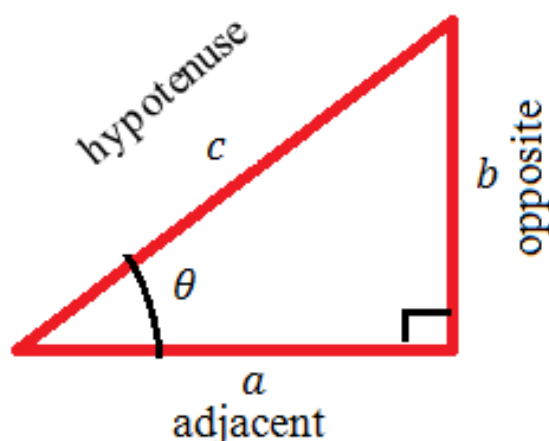


Trigonometric functions

There are six basic trigonometric functions used in Trigonometry. These functions are trigonometric ratios. The six basic trigonometric functions are sine function, cosine function, secant function, co-secant function, tangent function, and co-tangent function. The trigonometric functions and identities are the ratio of sides of a right-angled triangle. The sides of a right triangle are the perpendicular side, hypotenuse, and base, which are used to calculate the sine, cosine, tangent, secant, cosecant, and cotangent values using trigonometric formulas.

A right triangle is a triangle with a right angle (90°)



For every angle θ in the triangle, there is the side of the triangle adjacent to it, the side opposite of it and the hypotenuse such that $a^2 + b^2 = c^2$.

For angle θ , the trigonometric functions are defined as follows:

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{c}$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{c}$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta} = \frac{\text{adj}}{\text{opp}} = \frac{a}{b}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}} = \frac{c}{a}$$

$$\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}} = \frac{c}{b}$$

Trigonometric Functions Values

The trigonometric functions have a domain θ , which is in degrees or radians. Some of the principal values of θ for the different trigonometric functions are presented below in a table.

θ degrees	0°	30°	45°	60°	90°	180°	270°	360°
θ radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin \theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
$\csc \theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	-1	∞
$\sec \theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	-1	∞	1
$\cot \theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	∞	0	∞

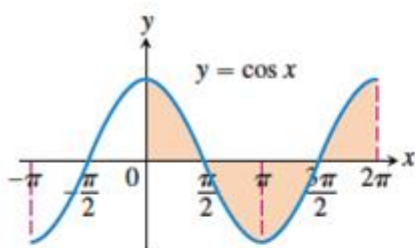
Trigonometric functions of negative angles

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos \theta \quad \text{and} \quad \tan(-\theta) = -\tan \theta$$

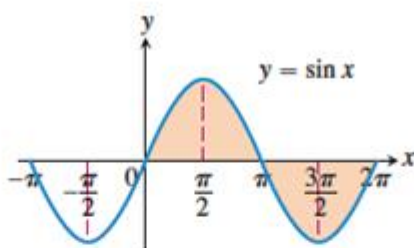
Some useful relationships among trigonometric functions

- $\sin^2 x + \cos^2 x = 1, \quad \sec^2 x - \tan^2 x = 1, \quad \csc^2 x - \cot^2 x = 1$
- $\sin 2x = 2 \sin x \cos x, \quad \cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$
- $\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$

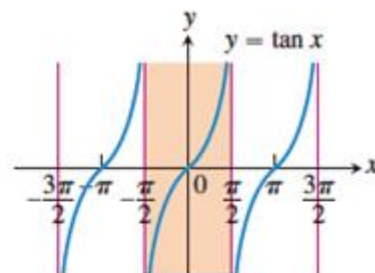
Graphs of Trigonometric Functions



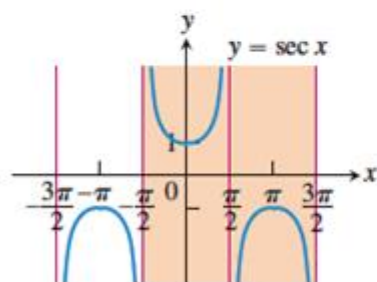
Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π



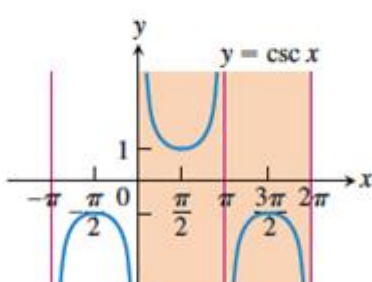
Domain: $-\infty < x < \infty$
Range: $-1 \leq y \leq 1$
Period: 2π



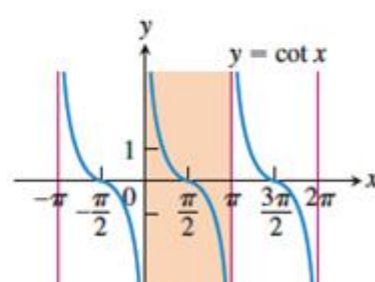
Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$
Range: $-\infty < y < \infty$
Period: π



Domain: $x \neq \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \dots$
Range: $y \leq -1$ and $y \geq 1$
Period: 2π



Domain: $x \neq 0, \pm \pi, \pm 2\pi, \dots$
Range: $y \leq -1$ and $y \geq 1$
Period: 2π



Domain: $x \neq 0, \pm \pi, \pm 2\pi, \dots$
Range: $-\infty < y < \infty$
Period: π

Derivatives of trigonometric functions

If u is a function x , the chain rule version of this differentiation rule is

$$1. \frac{d}{dx}(\sin u) = \cos u \cdot \frac{du}{dx}$$

$$2. \frac{d}{dx}(\cos u) = -\sin u \cdot \frac{du}{dx}$$

$$3. \frac{d}{dx}(\tan u) = \sec^2 u \cdot \frac{du}{dx}$$

$$4. \frac{d}{dx}(\cot u) = -\csc^2 u \cdot \frac{du}{dx}$$

$$5. \frac{d}{dx}(\sec u) = \sec u \tan u \cdot \frac{du}{dx}$$

$$6. \frac{d}{dx}(\csc u) = -\csc u \cot u \cdot \frac{du}{dx}$$

Example 1: Find the derivatives of the functions

1. $y = \sin^2 x \Rightarrow y = (\sin x)^2 \Rightarrow \frac{dy}{dx} = 2 \sin x \cos x = \sin 2x$
2. $y = \cos(x^2) \Rightarrow \frac{dy}{dx} = -2x \sin(x^2)$
3. $y = \tan \sqrt{x} \Rightarrow \frac{dy}{dx} = \sec^2 \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{\sec^2 \sqrt{x}}{2\sqrt{x}}$
4. $y = x^2 \sec 3x \Rightarrow \frac{dy}{dx} = 3x^2 \sec 3x \tan 3x + 2x \sec 3x$
 $= x \sec 3x (2 + 3x \tan 3x)$
5. $y = \sqrt{\sin 2x} \Rightarrow y = (\sin 2x)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} (\sin 2x)^{-1/2} \times \cos 2x \times 2$
 $= \frac{\cos 2x}{\sqrt{\sin 2x}}$

Example 2: If $y = \tan 2t$ and $x = \sec 2t$ show that $\frac{dy}{dx} = \csc 2t$

$$\begin{aligned} \frac{dy}{dt} &= 2 \sec^2 2t, & \frac{dx}{dt} &= 2 \sec 2t \tan 2t \\ \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} = 2 \sec^2 2t \times \frac{1}{2 \sec 2t \tan 2t} = \frac{\sec 2t}{\tan 2t} \\ &= \frac{1}{\frac{\cos 2t}{\sin 2t}} = \frac{1}{\sin 2t} = \csc 2t \end{aligned}$$

Example 3: If $y = \theta - \cos \theta$ and $x = \theta + \cos \theta$; $(0 \leq \theta \leq \frac{\pi}{2})$ show that $\frac{dy}{dx}$

$$= (\sec \theta + \tan \theta)^2$$

$$\frac{dy}{d\theta} = 1 + \sin \theta \quad \text{and} \quad \frac{dx}{d\theta} = 1 - \sin \theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta}$$

$$\frac{dy}{dx} = \frac{1 + \sin \theta}{1 - \sin \theta} \times \frac{1 + \sin \theta}{1 + \sin \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{1 - \sin^2 \theta} = \frac{1 + 2 \sin \theta + \sin^2 \theta}{\cos^2 \theta}$$

$$\frac{dy}{dx} = \frac{1}{\cos^2 \theta} + \frac{2 \sin \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \sec^2 \theta + 2 \sec \theta \tan \theta + \tan^2 \theta = (\sec \theta + \tan \theta)^2$$

Inverse trigonometric functions

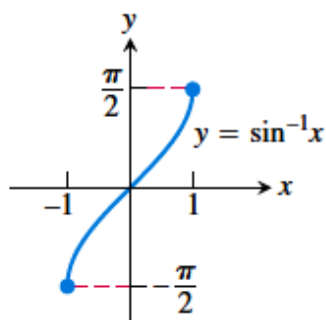
The inverse trigonometric functions are defined to be the inverses of particular parts of the trigonometric functions; parts that do have inverses. The inverse sine function, denoted by $\sin^{-1} x$ (some books use the notation $\arcsin(x)$), is defined to be the inverse of the restricted sine function. A similar idea holds for all the other inverse trigonometric functions. It is important here to note that in this case the “ -1 ” is not an

exponent and so, $\sin^{-1} x \neq \frac{1}{\sin x}$

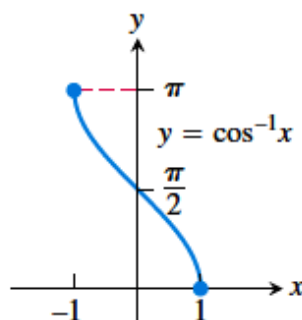
In inverse trigonometric functions the “ -1 ” looks like an exponent but it isn't, it is simply a notation that we use to denote the fact that we're dealing with an inverse trigonometric function. It is a notation that we use in this case to denote inverse trigonometric functions. If we had really wanted exponentiation to denote 1 over sine, we would use the following:

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

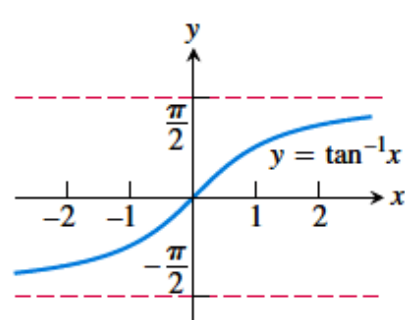
Domain: $-1 \leq x \leq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$



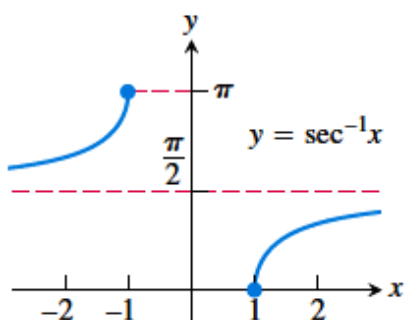
Domain: $-1 \leq x \leq 1$
Range: $0 \leq y \leq \pi$



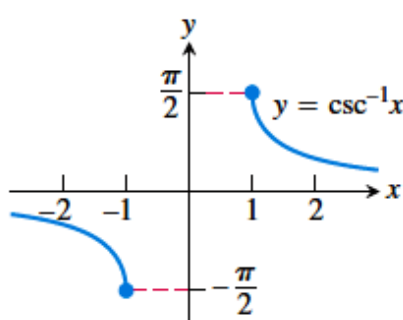
Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



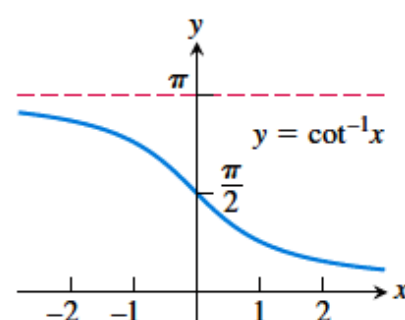
Domain: $x \leq -1$ or $x \geq 1$
Range: $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$



Domain: $x \leq -1$ or $x \geq 1$
Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$



Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



Derivatives of inverse trigonometric functions

Let u be a function x , the derivatives of inverse trigonometric functions are:

$$\begin{aligned} 1. \frac{d}{dx} (\sin^{-1} u) &= \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} & 2. \frac{d}{dx} (\cos^{-1} u) &= \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \\ 3. \frac{d}{dx} (\tan^{-1} u) &= \frac{1}{1+u^2} \cdot \frac{du}{dx} & 4. \frac{d}{dx} (\cot^{-1} u) &= \frac{-1}{1+u^2} \cdot \frac{du}{dx} \\ 5. \frac{d}{dx} (\sec^{-1} u) &= \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx} & 6. \frac{d}{dx} (\csc^{-1} u) &= \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx} \end{aligned}$$

Example 4: Find the derivative for

$$1. y = \sin^{-1} 2x \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(2x)^2}} \times 2 = \frac{2}{\sqrt{1-4x^2}}$$

$$\begin{aligned} 2. y &= 3x \cos^{-1} 3x - \sqrt{1-9x^2} \\ \frac{dy}{dx} &= 3x \times \frac{-1}{\sqrt{1-(3x)^2}} \times 3 + 3 \cos^{-1} 3x - \frac{-18x}{2\sqrt{1-9x^2}} \\ &= \frac{-9x}{\sqrt{1-9x^2}} + 3 \cos^{-1} 3x + \frac{9x}{\sqrt{1-9x^2}} = 3 \cos^{-1} 3x \end{aligned}$$

$$\begin{aligned} 3. y &= 2\sqrt{x} \tan^{-1} \sqrt{x} \\ \frac{dy}{dx} &= 2\sqrt{x} \times \frac{1}{1+(\sqrt{x})^2} \times \frac{1}{2\sqrt{x}} + 2 \tan^{-1} \sqrt{x} \times \frac{1}{2\sqrt{x}} = \frac{1}{1+x} + \frac{\tan^{-1} \sqrt{x}}{\sqrt{x}} \end{aligned}$$

Exercises

Find derivative in each of the following problems(1 – 4)

- $y = \sec^2 2x$
- $y = x^2 \sin x + 2x \cos x - 2 \sin x$
- $y = \sqrt{x^2 - 1} - \sec^{-1} x$
- $y = 2x \cos^{-1} \sqrt{x} + \sin^{-1} \sqrt{x} - 2\sqrt{x - x^2}$
- If $y = 1 - \sin \theta$ and $x = \theta - \sin \theta$ find $\frac{dy}{dx}$
- If $y = \sec^{-1} t$ and $x = \sqrt{t^2 - 1}$ find $\frac{dy}{dx}$