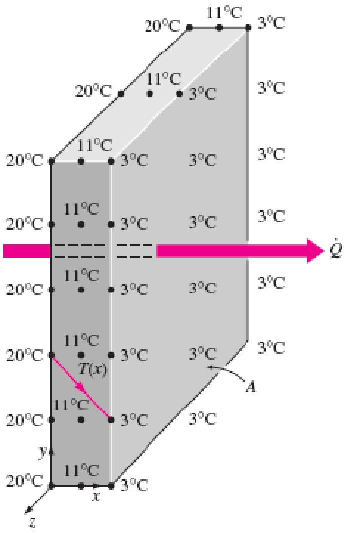
1-D plane wall



Energy balance

Rate of heat transfer into the wall - Rate of heat transfer out of the wall = Rate of change of energy of the wall

$$\dot{Q}_{in} - \dot{Q}_{out} = \frac{dE_{wall}}{dt}$$

$$\frac{dE_{wall}}{dt} = 0 \quad \text{for steady operation}$$

Therefore, the rate of heat transfer into the wall must be equal to the rate of heat transfer out of it. In other words, the rate of heat transfer through the wall must be constant, $Q_{cond,\,wall}$ constant.

Fourier's law of heat conduction for the wall $\dot{Q}_{cond,wall} = -kA\frac{dT}{dx}$

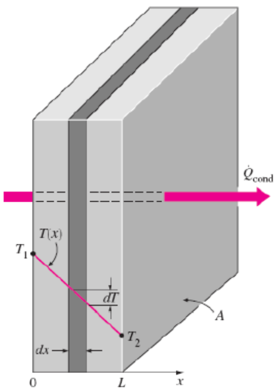
$$\int_{x=0}^{L} \dot{Q}_{cond,wall}\,dx = -\int_{T=T_1}^{T_2} kA\,dT$$

constant

Temp profile

$$\dot{Q}_{cond,wall} = kA\frac{T_1 - T_2}{L} \quad \text{(W)}$$

The rate of heat conduction through a plane wall is proportional to the average thermal conductivity, the wall area, and the temperature difference, but is inversely proportional to the wall thickness



1 D steady state heat conduction equation $\frac{d}{dx}(k\frac{dT}{dx}) = 0$

Integrate the above equation twice $T(x) = C_1x + C_2$

Boundary conditions $T(0) = T_{s,1}$ and $T(L) = T_{s,2}$

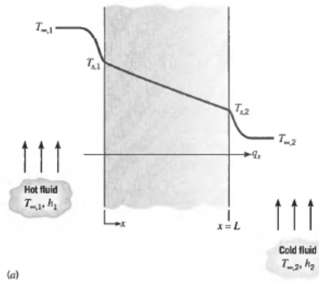
Apply the condition at x = 0 and L

$$T_{s,1} = C_2$$

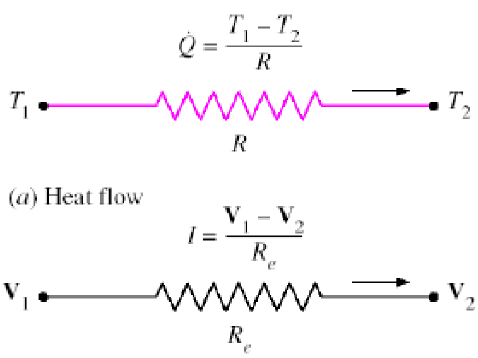
$$T_{s,2} = C_1L + C_2 = C_1L + T_{s,1}$$

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

$$T(x) = \frac{T_{s,2} - T_{s,1}}{L}x + T_{s,1}$$



Thermal Resistance Concept



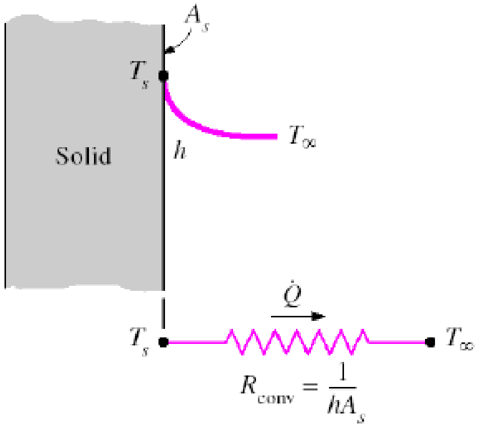
(a) Heat flow

(b) Electric current flow

$$\dot{Q}_{cond,wall} = \frac{T_1 - T_2}{R_{wall}} \quad \text{(W)}$$

$$R_{wall} = \frac{L}{kA} \quad \text{(°C/W)}$$

Convection Resistance



$$\dot{Q}_{convection} = hA_s(T_s - T_\infty)$$

$$\dot{Q}_{convection} = \frac{T_s - T_\infty}{R_{convection}} \quad \text{(W)}$$

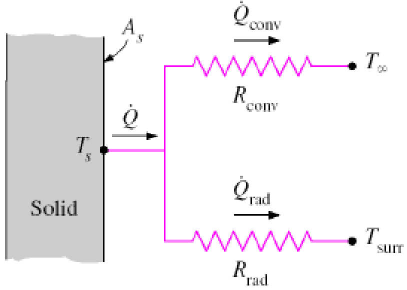
$$R_{convection} = \frac{1}{hA_s} \quad (^oC/W)$$

Radiation Resistance

$$\dot{Q}_{rad} = \varepsilon\sigma A_s(T_s^4 - T_{surr}^4) = h_{rad}A_s(T_s - T_{surr}) = \frac{T_s - T_{surr}}{R_{rad}} \quad \text{(W)}$$

$$R_{rad} = \frac{1}{h_{rad}A_s} \quad \text{(K/W)}$$

Schematic for convection and radiation resistances at a surface.



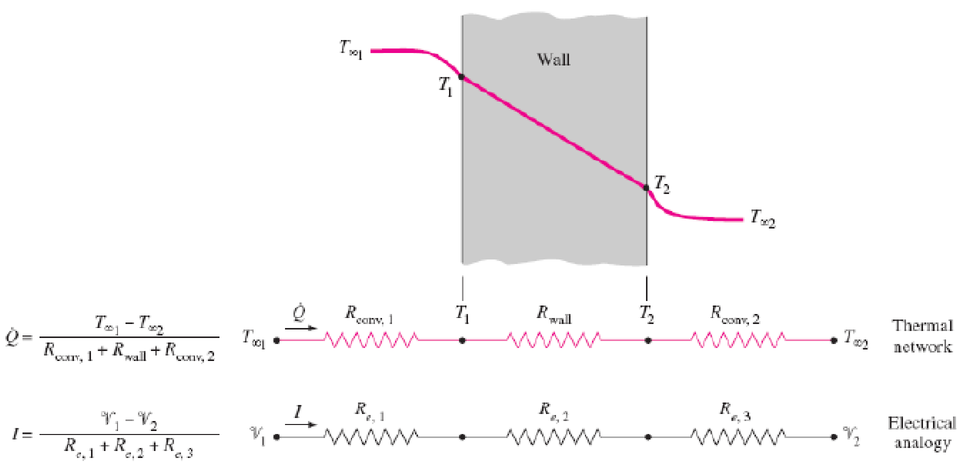
$$\dot{Q} = \dot{Q}_{conv} + \dot{Q}_{rad}$$

Combined convection and radiation

$$h_{rad} = \frac{\dot{Q}_{rad}}{A_s(T_s - T_{surr})} = \varepsilon\sigma(T_s^2 + T_{surr}^2)(T_s + T_{surr}) \quad (W/m^2K)$$

$$h_{combined} = h_{conv} + h_{rad} \quad (W/m^2K)$$

Possible when $T_\infty = T_{surr}$



The thermal resistance network for heat transfer through a plane wall subjected to convection on both sides, and the electrical analogy.

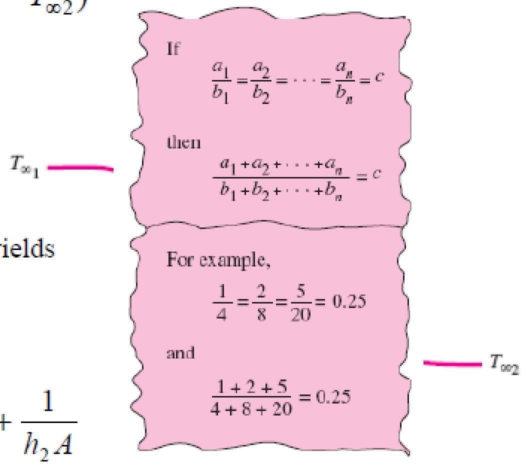
Network subjected to convection on both sides

Rate of heat convection into the wall = Rate of heat conduction through the wall = Rate of heat convection from the wall

$$\dot{Q} = h_1 A (T_{\infty 1} - T_1) = kA\frac{T_1 - T_2}{L} = h_2 A (T_2 - T_{\infty 2})$$

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{\infty 2}}{1/h_2 A}$$

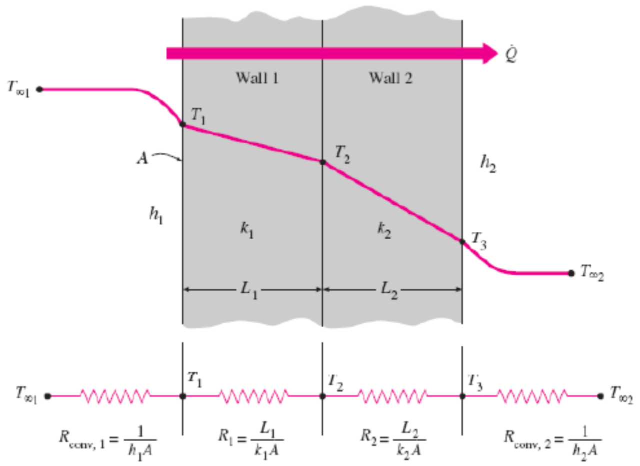
$$= \frac{T_{\infty 1} - T_1}{R_{conv,1}} = \frac{T_1 - T_2}{R_{wall}} = \frac{T_2 - T_{\infty 2}}{R_{conv,2}}$$



Adding the numerators and denominators yields

$$\dot{Q} = \frac{T_{\infty 1} - T_{\infty 2}}{R_{total}} \qquad \text{(W)}$$

$$R_{total} = R_{conv,1} + R_{wall} + R_{conv,2} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$



It is sometimes convenient to express heat transfer through a medium in an analogous manner to Newton's law of cooling as

$$\dot{Q} = UA\Delta T \qquad \text{(W)}$$

$$\dot{Q} = \frac{\Delta T}{R_{total}}$$

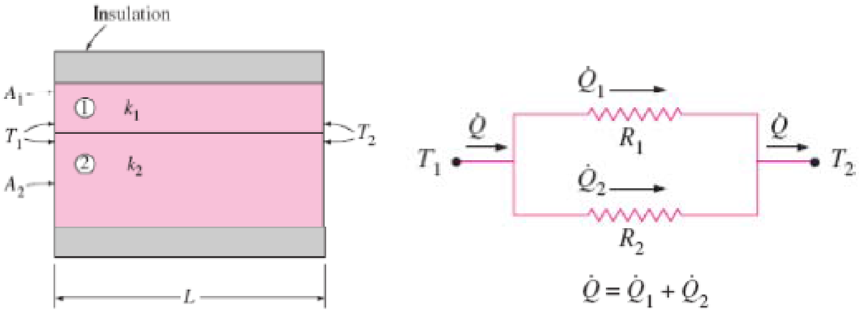
$$UA = \frac{1}{R_{total}}$$

The surface temperature of the wall can be determined using the thermal resistance concept, but by taking the surface at which the temperature is to be determined as one of the terminal surfaces.

$$\dot{Q} = \frac{T_{\infty 1} - T_1}{R_{conv,1}} = \frac{T_{\infty 1} - T_1}{1/h_1 A}$$

Known ($\dot{Q}$)

Two parallel layers



$$\dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$R_1 = \frac{L_1}{k_1 A_1'} \qquad R_2 = \frac{L_2}{k_2 A_2'}$$

$$\dot{Q} = \frac{T_1 - T_2}{R_{total}}$$

where $\frac{1}{R_{total}} = \frac{1}{R_1} + \frac{1}{R_2} \longrightarrow R_{total} = \frac{R_1 R_2}{R_1 + R_2}$

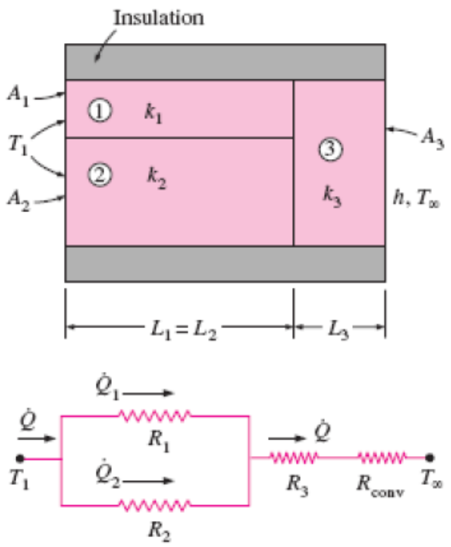
Combined series-parallel

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{total}}$$

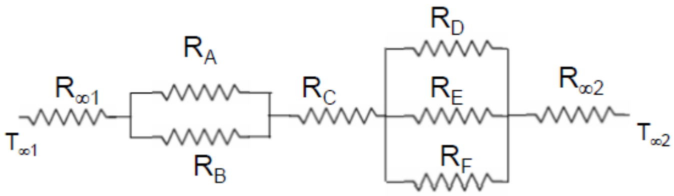
$$R_{total} = R_{12} + R_3 + R_{conv} = \frac{R_1 R_2}{R_1 + R_2} + R_3 + R_{conv}$$

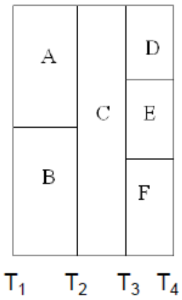
$$R_1 = \frac{L_1}{k_1 A_1'} \qquad R_2 = \frac{L_2}{k_2 A_2'} \qquad R_3 = \frac{L_3}{k_3 A_3'}$$

$$R_{conv} = \frac{1}{hA_3}$$



Series and parallel composite wall and its thermal circuit





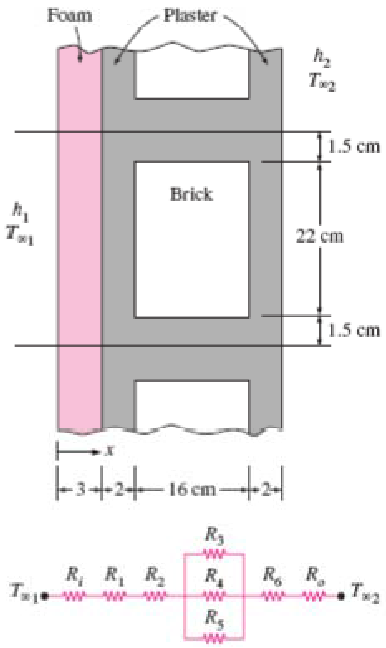
$$\sum R = R_{\infty 1} + \frac{1}{\left(\frac{1}{R_A} + \frac{1}{R_B}\right)} + R_C + \frac{1}{\left(\frac{1}{R_D} + \frac{1}{R_E} + \frac{1}{R_F}\right)} + R_{\infty 2}$$

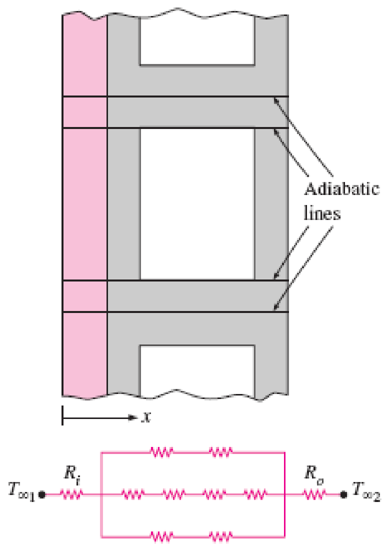
$$\dot{Q} = UA \ \Delta T \qquad \text{(W)}$$

where U is the overall heat transfer coefficient

$$UA = \frac{1}{R_{total}}$$

Complex multi-dimensional problems as 1-D problems





Heat conduction in cylinder

$$\dot{Q}_{cond,cyl} = -kA\frac{dT}{dr} \qquad A = 2\pi rL$$

$$\int_{r=r_1}^{r_2} \frac{\dot{Q}_{cond,cyl}}{A} dr = -\int_{T=T_1}^{T_2} kdT$$

Substituting $A = 2\pi rL$ and performing the integrations give

$$\dot{Q}_{cond,cyl} = 2\pi Lk\frac{T_1 - T_2}{\ln(r_2/r_1)}$$

$\dot{Q}_{cond,cyl}$ =constant at steady state

$$\dot{Q}_{cond,cyl} = \frac{T_1 - T_2}{R_{cyl}} \qquad (W)$$

$$R_{cyl} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{outer radius/inner radius})}{2\pi(\text{length})(\text{thermal conductivity})}$$

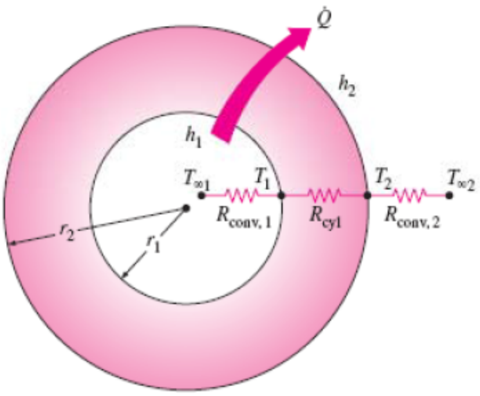
Heat conduction in sphere

For sphere

$$\dot{Q}_{cond,sphere} = \frac{T_1 - T_2}{R_{sph}}$$

$$R_{sph} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{outer radius - inner radius}}{4\pi(\text{outer radius})(\text{inner radius})(\text{thermal conductivity})}$$

Resistance Network



$R_{total} = R_{conv,1} + R_{cyl} + R_{conv,2}$

The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.

cylindrical

$$R_{total} = R_{conv,1} + R_{cond} + R_{conv,2}$$

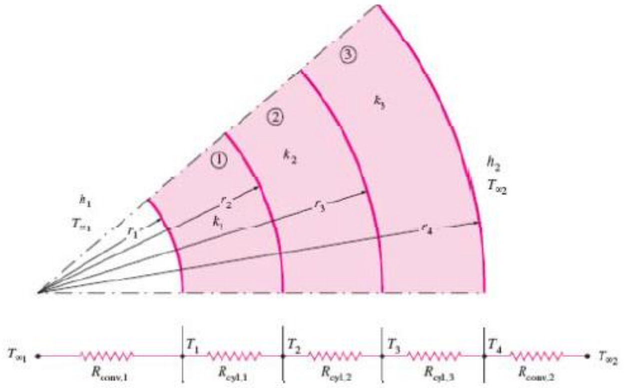
$$= \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi Lk} + \frac{1}{(2\pi r_2 L)h_2}$$

spherical

$$R_{total} = R_{conv,1} + R_{sph} + R_{conv,2}$$

$$= \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2}$$

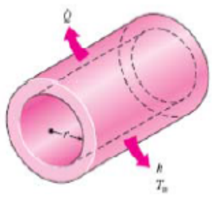
Multilayered cylinder



$$R_{total} = R_{conv,1} + R_{cyl,1} + R_{cyl,2} + R_{cyl,3} + R_{conv,2}$$

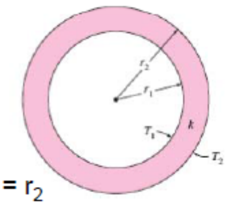
$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi L k_1} + \frac{\ln(r_3/r_2)}{2\pi L k_2} + \frac{\ln(r_4/r_3)}{2\pi L k_3} + \frac{1}{h_2 A_4}$$

Radial heat conduction through cylindrical systems



$$\frac{1}{r}\frac{\partial}{\partial r}\left(k.r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k.r\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k.\frac{\partial T}{\partial z}\right) + \dot{g} = \rho C\frac{\partial T}{\partial t}$$

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0$$



Integrating the above equation twice, $T=C_1 \ln r + C_2$

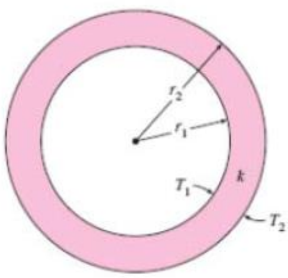
Subject to the boundary conditions, $T=T_1$ at $r = r_1$ and $T=T_2$ at $r = r_2$

$$T = \frac{T_2 - T_1}{\ln\left(\frac{r_2}{r_1}\right)}\ln(r) + \frac{T_1 \ln r_2 - T_2 \ln r_1}{\ln\left(\frac{r_2}{r_1}\right)}$$

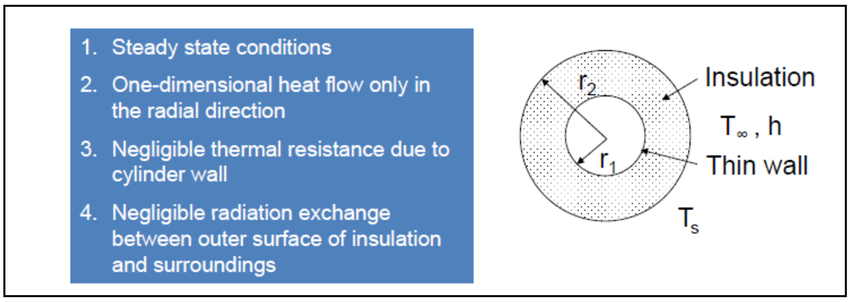
$$Q = -kA_r \frac{dT}{dr}\Big|_{r=r_1} = -k.2\pi r_1 L.\frac{C_1}{r_1}$$

$$Q = -k.2\pi r_1 L.(T_2 - T_1).\frac{1}{r_1 \ln\left(\frac{r_2}{r_1}\right)}$$

$$= \frac{2\pi k L(T_1 - T_2)}{\ln\left(\frac{r_2}{r_1}\right)}$$



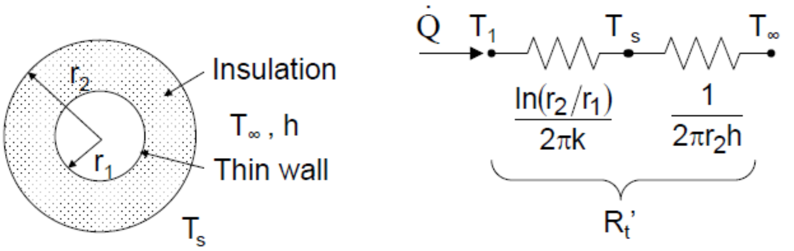
Critical Radius of Insulation

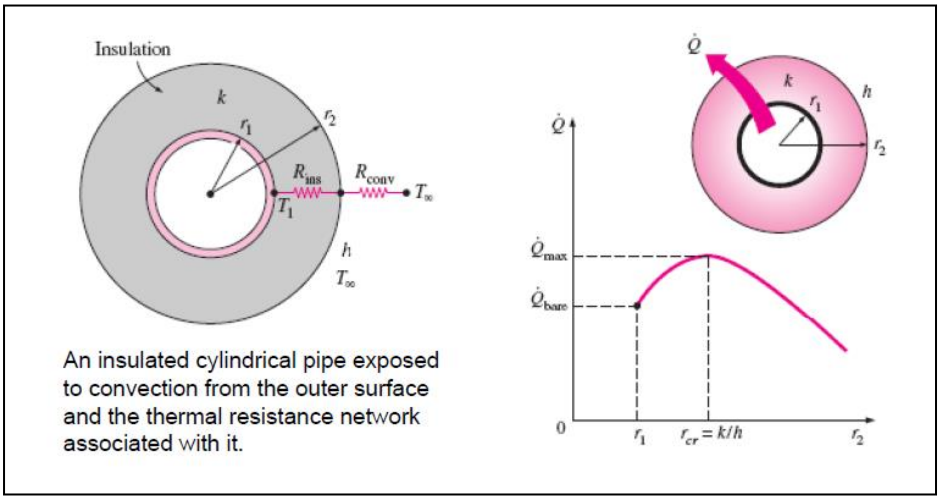


Critical Radius of Insulation

• Practically, it turns out that adding insulation in cylindrical and spherical exposed walls can initially cause the thermal resistance to decrease, thereby increasing the heat transfer rate because the outside area for convection heat transfer is getting larger. At some critical thickness, r_{cr}, the thermal resistance increases again and consequently the heat transfer is reduced.

• To find an expression for r_{cr}, consider the thermal circuit below for an insulated cylindrical wall with thermal conductivity k:





An insulated cylindrical pipe exposed to convection from the outer surface and the thermal resistance network associated with it.

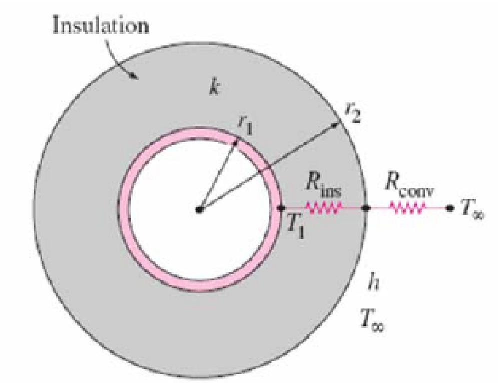
- To find r_{cr}, set the overall thermal resistance $dR_t'/dr = 0$ and solve for r:

$$R_t' = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h} \qquad r_i = \text{inner radius}$$

$$\frac{dR_t'}{dr} = \frac{1}{2\pi k r} - \frac{1}{2\pi r^2 h} = 0$$

$$r = r_{cr} = \frac{k}{h} \qquad \text{Similarly for a sphere} \qquad r_{cr} = \frac{2k}{h}$$

- For insulation thickness less that r_{cr} the heat loss increases with increasing r and for insulation thickness greater that r_{cr} the heat loss decreases with increasing r
- If k = 0.03 W/(m·K) and h = 10 W/(m²·K):
  - cylinder $r_{cr} = \frac{k}{h} = \frac{0.03 W/(m\cdot K)}{10 W/(m^2\cdot K)} = 0.003 m = 3 mm$
  - sphere $r_{cr} = \frac{2k}{h} = 6 mm$



of r_1, h and k are constant

Total thermal resistance per unit length

$$R_{total} = \frac{\ln\left(\frac{r_2}{r_1}\right)}{2\pi k} + \frac{1}{2\pi r_2 h}$$

Heat transfer per unit length $\frac{\dot{Q}}{L} = \frac{T_\infty - T_i}{R_{total}}$

Optimum thickness is associated with r_2, $\frac{dR_{total}}{dr_2} = 0$

$$\frac{1}{2\pi k r_2} - \frac{1}{2\pi r_2^2 h} = 0 \qquad r_2 = \frac{k}{h}$$

To see the condition maximizes or minimizes the total resistance

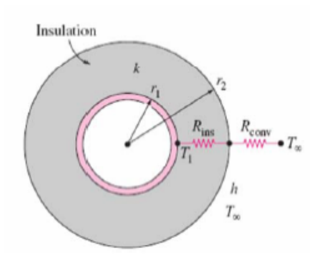
$$\frac{d^2R_{total}}{dr_2^2} = -\frac{1}{2\pi k r_2^2} + \frac{1}{\pi r_2^3 h}$$

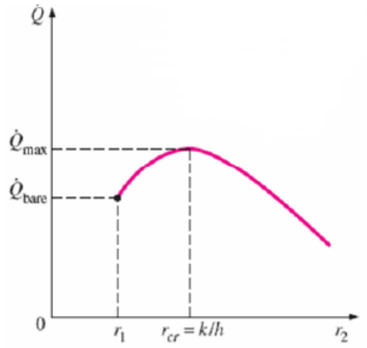
At r_2=k/h

$$\frac{d^2R_{total}}{dr_2^2} = \frac{1}{\pi(k/h)^2}\left(\frac{1}{k} - \frac{1}{2k}\right) = \frac{1}{2\pi k^3/h^2} > 0$$

Always positive, total resistance at *k/h* is minimum

$$r_{cr,cylinder} = \frac{k}{h} \qquad (m)$$





$$r_{cr,max} = \frac{k_{max,\ insulation}}{h_{min}} \approx \frac{0.05\left(W/m\,°C\right)}{5\left(W/m^{2}°C\right)} = 0.01m = 1cm$$

We can insulate hot water pipes and steam lines without worrying the critical radius of insulation

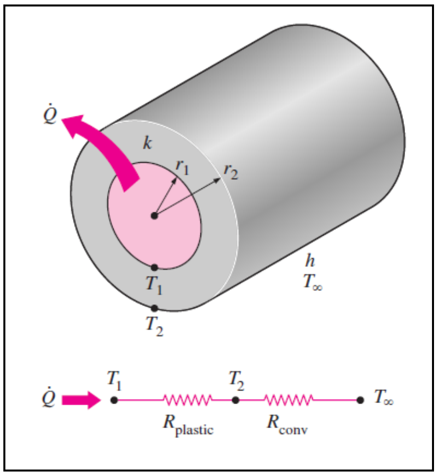
Insulation of electric wires:

-Radius of electric wires may be smaller than the critical radius

-Addition of insulation material increases heat transfer

Critical radius of insulation for spherical shell: $r_{cr,sphere} = \frac{2k}{h}$

Ex1: A 3-mm-diameter and 5-m-long electric wire is tightly wrapped with a 2-mm thick plastic cover whose thermal conductivity is k = 0.15 W/m · °C. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at T_∞ = 30°C with a heat transfer coefficient of h = 12 W/m^2 · °C, determine
the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

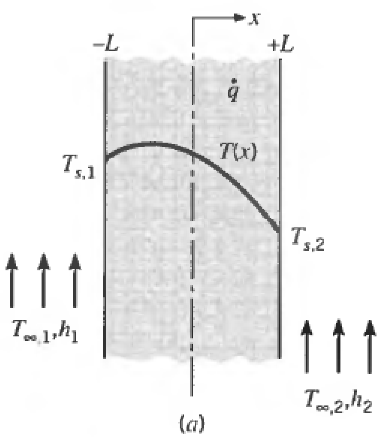


1D Conduction with Heat Generation

The Plane Wall

$$\frac{d^2T}{dx^2} + \frac{\dot{q}}{k} = 0$$

$$T(x) = -\frac{\dot{q}}{2k}x^2 + C_1 x + C_2$$



Boundary conditions:

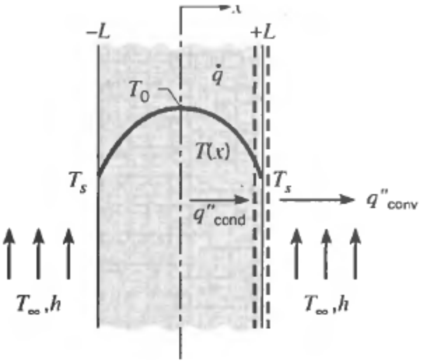
$$T(-L) = T_{s,1} \qquad C_1 = \frac{T_{s,2} - T_{s,1}}{2L}$$

$$T(L) = T_{s,2} \qquad C_2 = \frac{\dot{q}}{2k}L^2 + \frac{T_{s,2} + T_{s,1}}{2}$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + \frac{T_{s,2} - T_{s,1}}{2}\frac{x}{L} + \frac{T_{s,2} + T_{s,1}}{2}$$

$$T_{s,1} = T_{s,2} \equiv T_s$$

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s$$



The maximum temperature exists at the midplane

$$T(0) \equiv T_o = \frac{\dot{q}L^2}{2k} + T_s \qquad \text{Put } x = 0$$

If the surface temperature of the heat generating body is unknown and the surrounding fluid temperature is T_∞

Using energy balance $-k\frac{dT}{dx}|_{x=L} = h(T_s - T_\infty)$

Find temperature gradient from the above Eq. at x = L

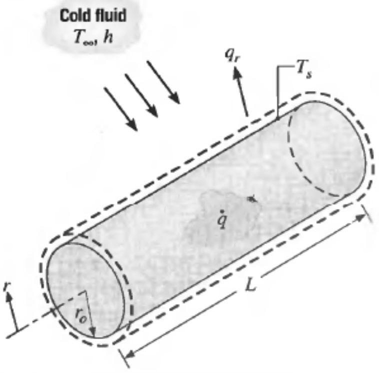
We can obtain the surface temperature $T_s = T_\infty + \frac{\dot{q}L}{h}$

Radial Systems

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)+\frac{\dot{q}}{k}=0$$

$$r\frac{dT}{dr}=-\frac{\dot{q}\,r^2}{2k}+C_1$$

$$T(r)=-\frac{\dot{q}\,r^2}{4k}+C_1\ln r+C_2$$



Conduction in a solid cylinder with uniform heat generation.

Boundary conditions:

$$\frac{dT}{dr}\Big|_{r=0}=0 \qquad T(r_o)=T_s$$

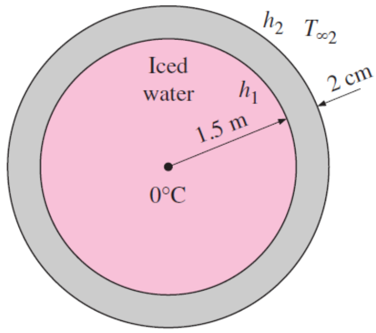
$$C_1=0 \qquad C_2=T_s+\frac{\dot{q}\,r_o^2}{4k}$$

$$T(r)=\frac{\dot{q}\,r_o^2}{4k}\left(1-\frac{r^2}{r_o^2}\right)+T_s$$

$$\dot{q}\left(\Pi r_o^2 L\right)=h(2\Pi r_o L)(T_s-T_\infty)$$

$$T_s=T_\infty+\frac{\dot{q}\,r_o}{2h}$$

Ex2: A 3-m internal diameter spherical tank made of 2-cm-thick stainless steel (k = 15 W/m · °C) is used to store iced water at $T_{\infty,1}$ = 0°C. The tank is located in a room whose temperature is $T_{\infty,2}$ = 22°C. The walls of the room are also at 22°C. The outer surface of the tank is black and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation.
The convection heat transfer coefficients at the inner and the outer surfaces of the tank are $h1$ = 80 W/m^2 · °C and h_2 = 10 W/m^2 · °C, respectively. Determine (*a*) the rate of heat transfer to the iced water in the tank and (*b*) the amount of ice at 0°C that melts during a 24-h period.
Take h_{rad} =5.34 , Noting that it takes 333.7 kJ of energy to melt 1 kg of ice at 0°C, (h_{if})



Ex3: Steam at $T_{\infty,1}$ =320°C flows in a cast iron pipe (k = 80 W/m · °C) whose inner and outer diameters are D_1 = 5 cm and D_2 = 5.5 cm, respectively. The pipe is covered with 3-cm-thick glass wool insulation with k = 0.05 W/m · °C. Heat is lost to the surroundings at $T_{\infty,2}$ =5°C by natural convection and radiation, with a combined heat transfer coefficient of h_2 = 18 W/m^2 · °C. Taking the heat transfer coefficient inside the pipe to be h_1 = 60 W/m^2 · °C, determine the rate of heat loss from the steam per unit length of the pipe. Also determine the temperature drops across the pipe shell and the insulation.