

Graphs:

A graph G consists of two things:

- (i) A set V whose elements are vertices, points or nodes.
- (ii) A set E of unordered pairs of distinct vertices called edges.

We denote such a graph by $G(V,E)$.

Vertices u and v are said to be adjacent if there is an edge $\{u,v\}$.

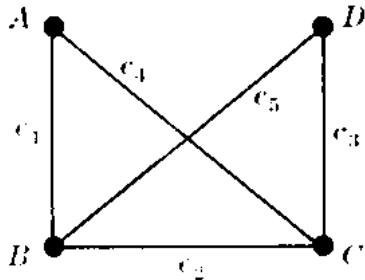
Graphs are the most useful model with computer science such as logical design, formal languages, communication network, compiler writing and retrieval.

$G(V,E)$

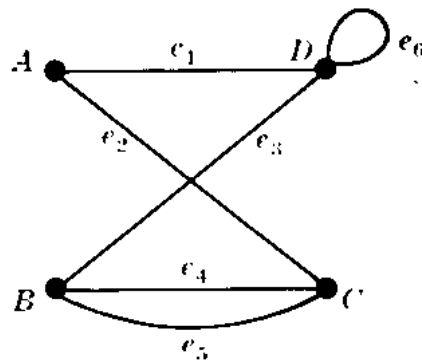
$V = \{V1, V2, V3, V4\}$

$E = \{e1, e2, e3, e4\}$

$E = \{(v1,v2),(v2,v3),(v3,v1),(v3,v4)\}$



(a) Graph



(b) Multigraph

For example we have in (a) the graph $G(V,E)$ where (i) V consists of four vertices A, B, C, D ; and (ii) E consists of five edges $e1 = \{A,B\}$, $e2 = \{B,C\}$, $e3 = \{C, D\}$, $e4 = \{A, C\}$, $e5 = \{B, D\}$.

The diagram in (b) is not a graph but a multigraph. The reason is that $e4$ and $e5$ are multiple edges, i.e. edges connecting the same endpoints, and $e6$ is a loop, i.e. an edge whose endpoints are the same vertex. The definition of a graph does not permit such multiple edges or loops.

Let $G(V,E)$ be a graph. Let V' be a subset of V and let E' be subset of E whose endpoints belong to V' . Then $G(V',E')$ is a graph and is called a subgraph of $G(V,E)$. If E' contains all the edges of E whose endpoints lie in V' , then $G(V',E')$ is called the subgraph generated by V' .

Degree :

The degree of a vertex v , written $\deg(v)$, is equal to the number of edges which are incident on v . since each edge is counted twice in counting the degrees of the vertices of a graph, we have the following result.

Theorem: The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

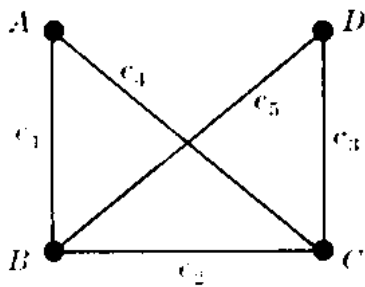
For example, in the figure (a) we have

$$\begin{aligned} \deg(A) &= 2, \\ \deg(B) &= 3, \\ \deg(C) &= 3, \\ \deg(D) &= 2 \end{aligned}$$

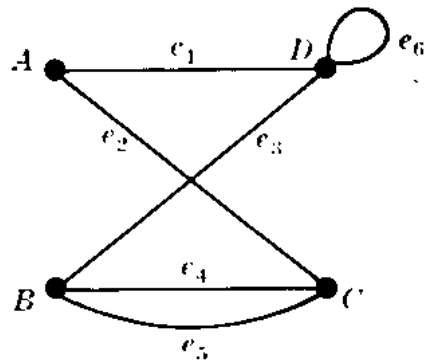
The sum of the degrees equals ten which, as expected, is twice the number of edges. A vertex is said to be **even** or **odd** according as its degree is an even or odd number. Thus A and D are even vertices whereas B and C are odd vertices.

This theorem also holds for multigraphs where a loop is counted twice towards the degree of its endpoint. For example, in Fig (b) we have $\deg(D) = 4$ since the edge e_6 is counted twice; hence D is an even vertex

A vertex of degree zero is called an isolated vertex.



(a) Graph



(b) Multigraph

Connectivity

A **walk** in a multigraph consists of an alternating sequence of vertices and edges of the form

$$v_0, e_1, v_1, e_2, v_2, \dots, e_{n-1}, v_{n-1}, e_n, v_n$$

Length of walk: is the number n of edges.

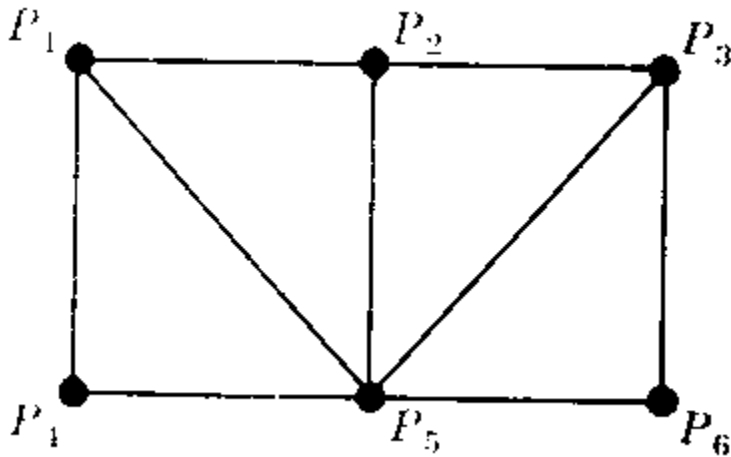
Closed walk: the walk is said to be closed if $v_0 = v_n$. Otherwise, we say that the walk is from v_0 to v_n .

Trail: is a walk in which all edges are distinct.

Path: is a walk in which all vertices are distinct.

Cycle: is a closed walk such that all vertices are distinct except $v_1 = v_n$

Example: Consider the following graph, then



(P4, P1, P2, P5, P1, P2, P3, P6)

is a walk from P4 to P6. It is not a trail since the edge {P1,P2} is used twice.

The sequence: (P4, P1, P5, P3, P5, P6)

Is not a walk since there is no edge {P2, P6}.

The sequence: (P4, P1, P5, P2, P3, P5, P6)

Is a trail since no edge is used twice; but it is not a path since the vertex P5 is used twice.

The sequence: (P4, P1, P5, P3, P6)

Is a path from P4 to P6.

The shortest path from P4 to P6 is (P4, P5, P6) which has length 2 (2 edges only)

The distance between vertices u & v $d(u,v)$ is the length of the shortest path

$d(P4,P6) = 2$

Connectivity, Connected Components

A graph G is connected if there is a path between any two of its vertices. The graph in Fig.(4) is connected, but the graph in Fig. (5) is not connected since, for example, there is no path between vertices D and E.

Suppose G is a graph. A connected subgraph H of G is called a connected component of G if H is not contained in any larger connected subgraph of G . It is clear that any graph G can be partitioned into its connected components. For example, the graph G in Fig. (5) has three connected components, the subgraphs induced by the vertex sets {A,C,D}, {E,F}, and {B}.

The vertex B in Fig. (5) is called an isolated vertex since B does not belong to any edge or, in other words, $\text{deg}(B) = 0$

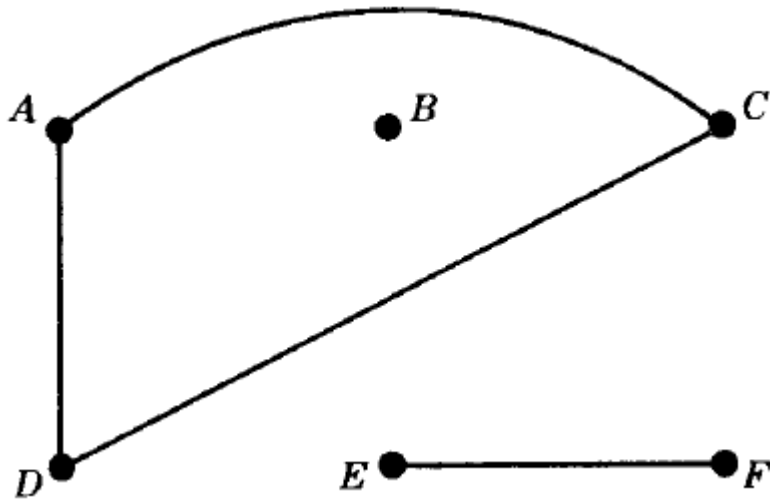


Figure (5)

Distance

Consider a connected graph G . The distance between vertices u and v in G , written $d(u,v)$, is the length of the shortest path between u and v . For example, in Fig. (6), $d(A,F) = 2$

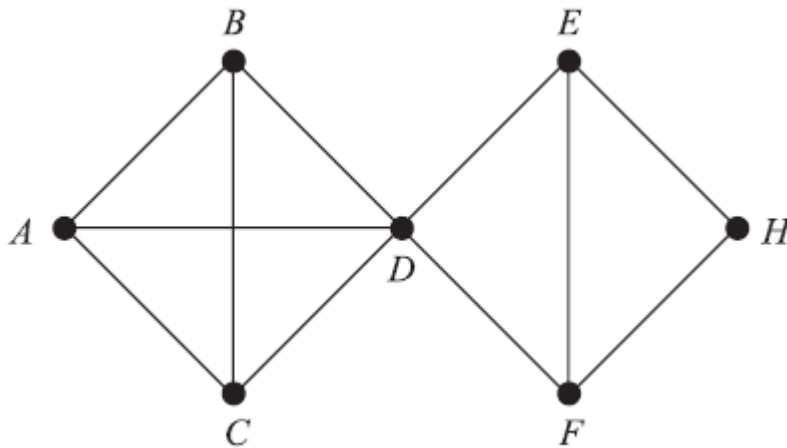


figure (6)

Cutpoints and Bridges

Let G be a connected graph. A vertex v in G is called a cutpoint if $G-v$ is disconnected. ($G-v$ is the graph obtained from G by deleting v and all edges containing v .) An edge e of G is called a bridge if $G - e$ is disconnected. ($G - e$ is the graph obtained from G by simply deleting the edge e .) In Fig. (6), the vertex D is a cutpoint and there are no bridges.

In Fig. (7), the edge $e = \{D,F\}$ is a bridge. (Its endpoints D and F are cutpoints.)

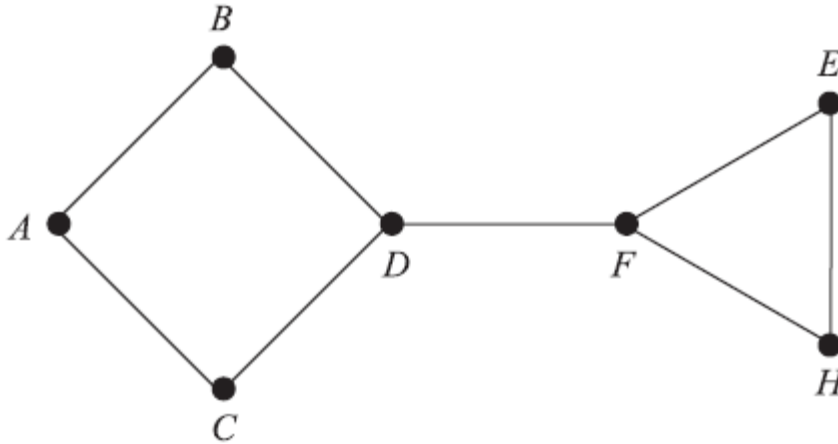
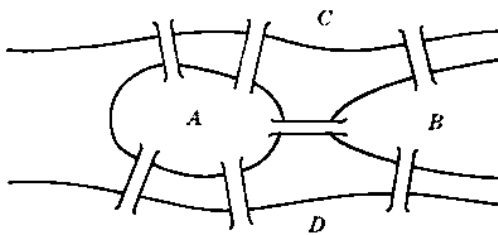


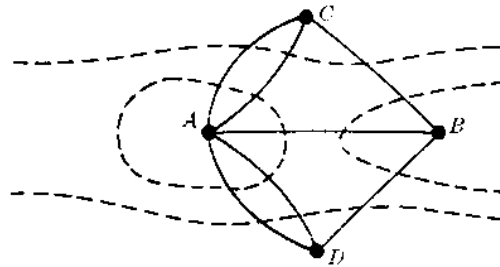
figure (7)

The Bridges of konigsberg, traversable multigraphs

The eighteenth-century East Prussian town of konigsberg included two islands and seven bridges. Question: beginning anywhere and ending anywhere, can a person walk through town crossing all seven bridges but not crossing any bridge twice? The people of Konigsberg wrote to the celebrated Swiss mathematician L. Euler about this question. Euler proved in 1736 that such a walk is impossible. He replaced the islands and two side of the river by points and the bridges by curves, obtaining Fig (b).

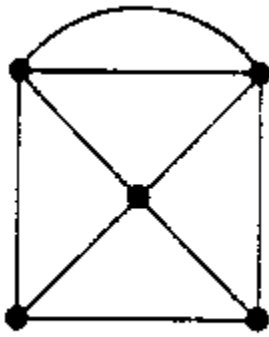


(a) Königsberg in 1736

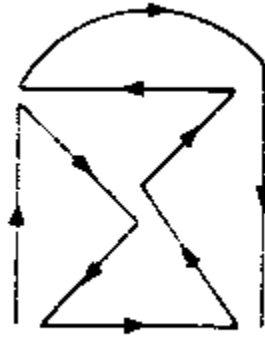


(b) Euler's graphical representation

Konigsberg graph is a multigraph
 A multigraph is said to traversable if it can be drawn without any breaks and without repeating any edge. That is if there is a walk includes all vertices and uses each edge exactly once. Such a walk must be a trail (no edge is used twice)



(a)



(b)

We now show how Euler proved that the konigsberg multigraph is not traversable and the walk in it is impossible. Suppose a multigraph is traversable and that a traversable trail does not begin or end at vertex P. thus the edges in the trail incident with P must appear in pairs, and so P is an even vertex. Therefore if a vertex Q is odd, the traversable trail must begin or end at Q. Consequently, a multigraph with more than two odd vertices cannot be traversable. Observe that the multigraph corresponding to the Konigsberg bridge problem has four odd vertices. Thus one cannot walk through Konigsberg so that each bridge is crossed exactly once.

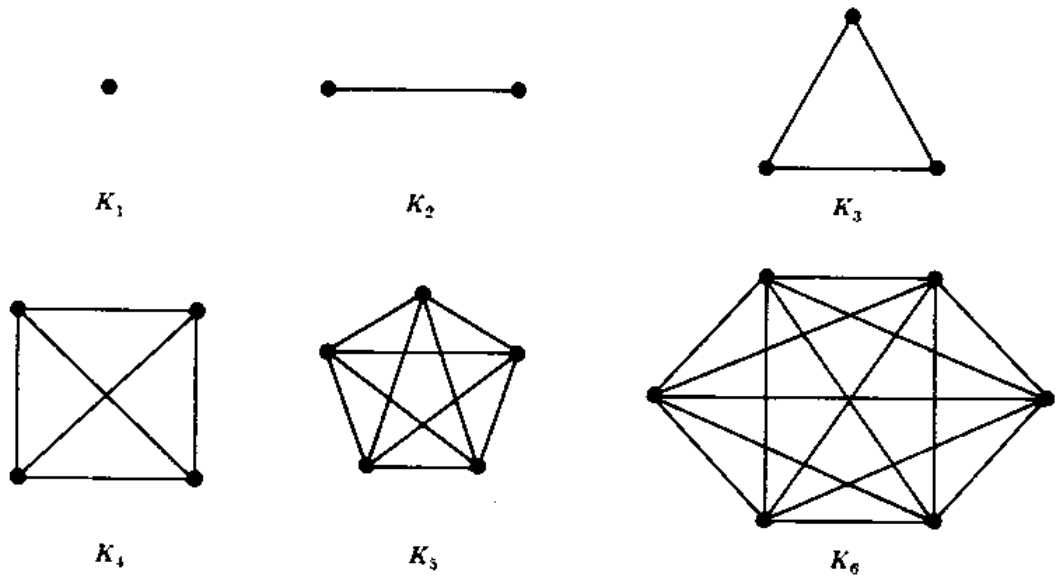
Theorem (Euler) : A finite connected graph is eulerian if and only if each vertex has even degree.

Corollary: Any finite connected graph with two odd vertices is traversable. A traversable trail may begin at either odd vertex and will end at the other odd vertex.

Special graph

1- Complete graph:

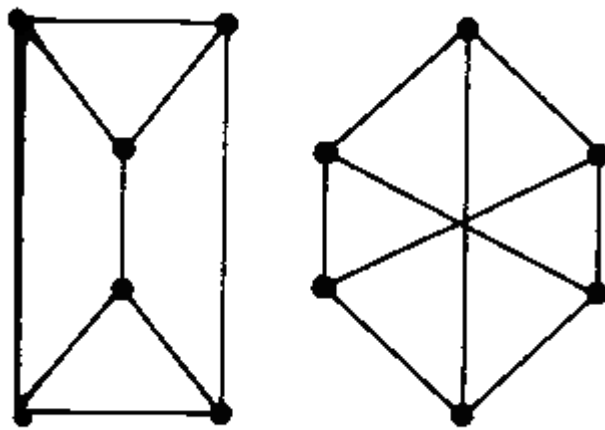
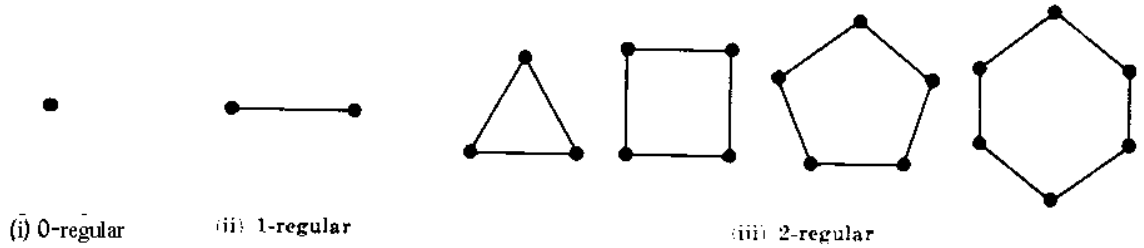
Each vertex is connected to every other vertex. The complete graph with n vertices is denoted by K_n



2- Regular Graph

Every vertex has the same degree. A graph G is regular of degree K or K -regular if every vertex has degree K .

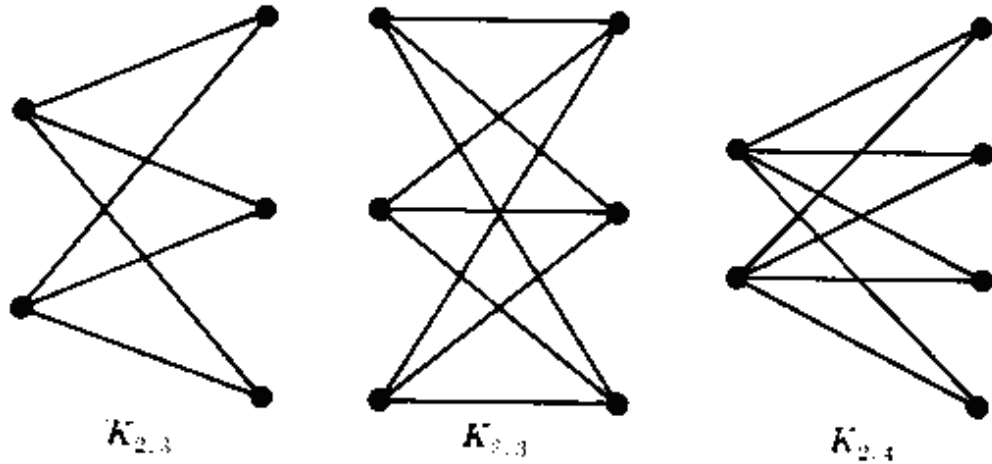
Example: 2-regular graph with every vertex has degree 2.



(vi) 3-regular

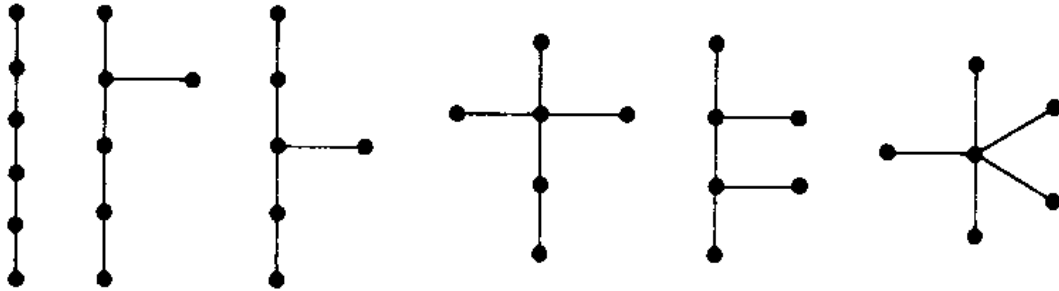
3- Bipartite graph :

Graph G is said to be bipartite if its vertices V can be partitioned into two subsets M and N such that each edge of G connects a vertex of M to a vertex of N .



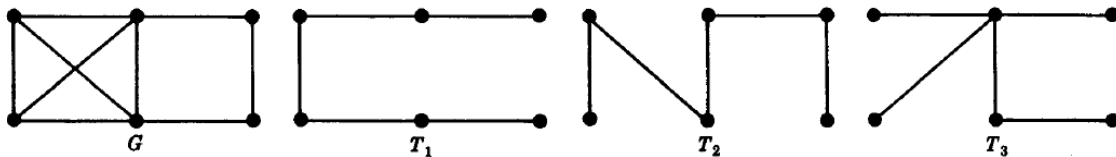
4 Tree graph:

A graph is said to be cycle-free or acyclic if it has no cycle. A connected graph with no cycle is called a tree



Spanning Trees

A subgraph T of a connected graph G is called a spanning tree of G if T is a tree and T includes all the vertices of G .



Minimum Spanning Trees

Suppose G is a connected weighted graph. That is, each edge of G is assigned a nonnegative number called the weight of the edge. Then any spanning tree T of G is assigned a total weight obtained by adding the weights of the edges in T . A minimal spanning tree of G is a spanning tree whose total weight is as small as possible.

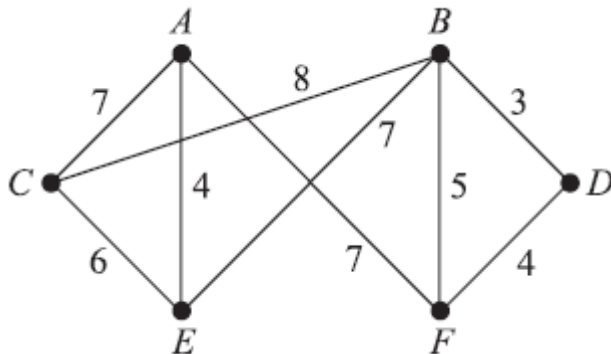
Algorithm 1 : The input is a connected weighted graph G with n vertices.

Step 1. Arrange the edges of G in the order of decreasing weights.

Step 2. Proceeding sequentially, delete each edge that does not disconnect the graph until $n - 1$ edges remain.

Step 3. Exit.

EXAMPLE: Find a minimal spanning tree of the weighted graph Q , Note that Q has six vertices, so a minimal spanning tree will have five edges.



First we order the edges by decreasing weights, and then we successively delete edges without disconnecting Q until five edges remain. This yields the following data:

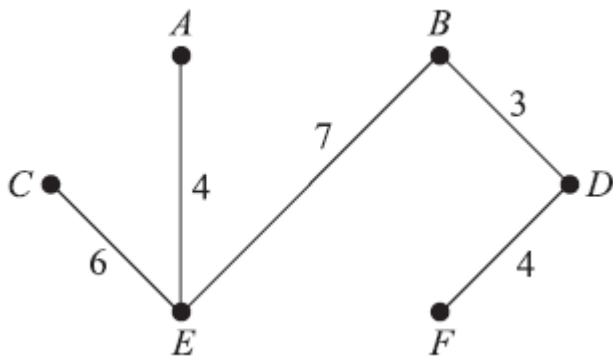
Edges: BC AF AC BE CE BF AE DF BD

Weight 8 7 7 7 6 5 4 4 3

Delete Yes Yes Yes No No Yes

Thus the minimal spanning tree of Q which is obtained contains the edges

BE, CE, AE, DF, BD The spanning tree has weight 24



Algorithm 2 (Kruskal): The input is a connected weighted graph G with n vertices.

Step 1. Arrange the edges of G in order of increasing weights.

Step 2. Starting only with the vertices of G and proceeding sequentially, add each edge which does not result in a cycle until $n - 1$ edges are added.

Step 3. Exit.

First we order the edges by increasing weights, and then we successively add edges without forming any cycles until five edges are included. This yields the following data:

Edges BD AE DF BF CE AC AF BE BC

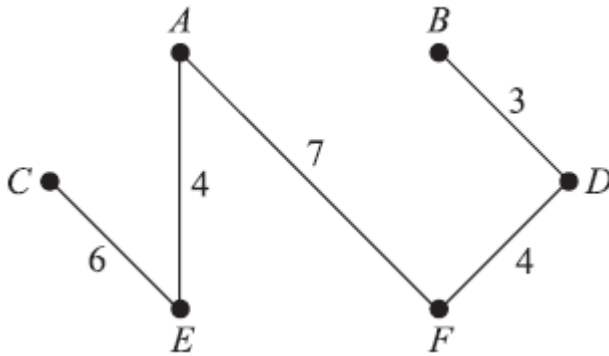
Weight 3 4 4 5 6 7 7 7 8

Add? Yes Yes Yes No Yes No Yes

Thus the minimal spanning tree of Q which is obtained contains the edges

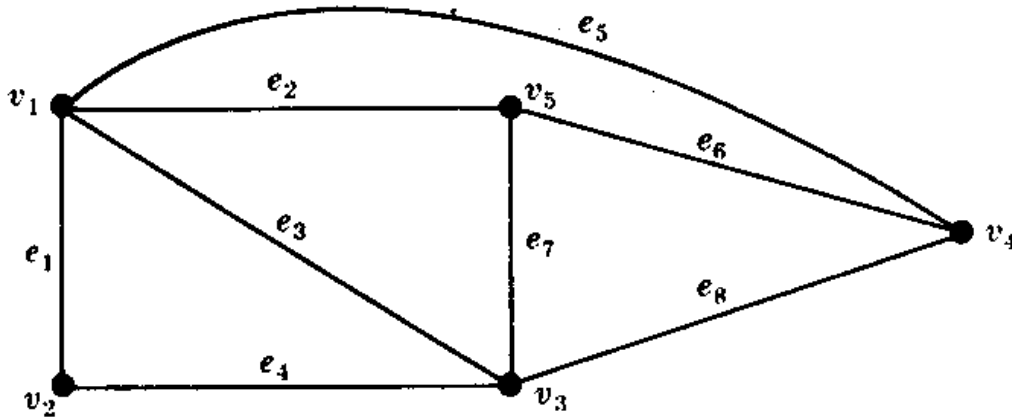
BD, AE, DF, CE, AF

Observe that this spanning tree is not the same as the one obtained using Algorithm 1 as expected it also has weight 24.



Matrices and Graphs:

Let G be a graph with vertices v_1, v_2, \dots, v_m and edges e_1, e_2, \dots, e_n . It is sometimes practical, especially for computational reasons, to represent G by a matrix. Note that the edges of G can be represented by an $n \times 2$ integer matrix B where each row of B denotes an edge of G , e.g. the row $(3,4)$ would denote the edge (v_3, v_4) . This edge matrix B does not completely describe G unless we are also given the number m of vertices of G . We do discuss two other widely used matrix representations of G .



$$B = \begin{pmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 3 \\ 2 & 3 \\ 1 & 4 \\ 4 & 5 \\ 3 & 5 \\ 3 & 4 \end{pmatrix}$$

(1) **Adjacency matrix.** Let $A = (a_{ij})$ be the $m \times m$ matrix defined by:

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge, i.e. if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Then A is called the adjacency matrix of G . Observe that $a_{ij} = a_{ji}$; hence A is a symmetric matrix.

$$A = \begin{matrix} & v_1 & v_2 & v_3 & v_4 & v_5 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

(2) **Incidence matrix.** Let $M = (m_{ij})$ be the $m \times n$ matrix defined by

$$m_{ij} = \begin{cases} 1 & \text{if the vertex } v_i \text{ is incident on the edge } e_j \\ 0 & \text{otherwise} \end{cases}$$

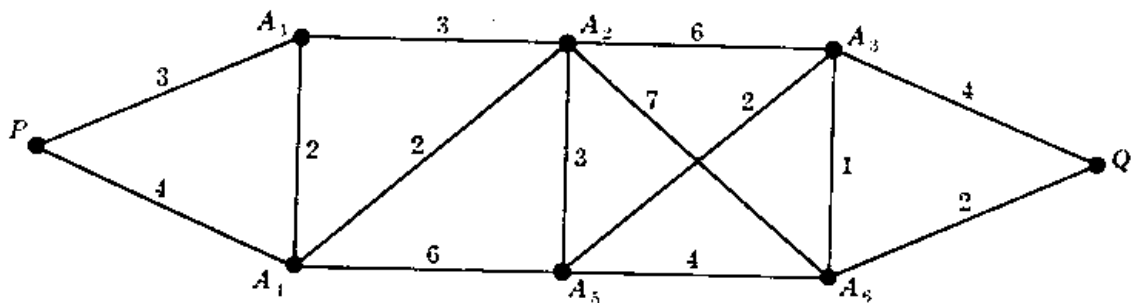
$$M = \begin{matrix} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{matrix} & \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{matrix}$$

Labeled graphs:

A graph G is called a labeled graph if its edges and/or vertices are assigned data. If each edge (e) is assigned a non negative number $L(e)$. Then $L(e)$ is called the weight or length of e .

One important problem in graph theory is to find a minimum path between two given points.

Example: find the minimum path between P & Q :



(P, A1, A2, A5, A3, A6, Q)

$$\sum_P^Q L(e) = 3 + 3 + 3 + 2 + 1 + 2 = 14$$

Another minimum path:

(P, A4, A2, A5, A3, A6, Q)

$$\sum_P^Q L(e) = 4 + 2 + 3 + 2 + 1 + 2 = 14$$