5.1. Joint Distribution

Let X and Y be random variables on a sample space S with respective image sets

$$X(S) = \{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad Y(S) = \{y_1, y_2, \ldots, y_m\}$$

We make the product set

$$X(S) \times Y(S) = \{(x_1, y_1), (x_1, y_2), \ldots, (x_n, y_m)\}$$

into a probability space by defining the *probability* of the ordered pair (x_i, y_j) to be $P(X = x_i, Y = y_j)$ which we write $h(x_i, y_j)$. This function h on $X(S) \times Y(S)$, i.e. defined by $h(x_i, y_j) = P(X = x_i, Y = y_j)$, is called the *joint distribution* or *joint probability function* of X and Y and is usually given in the form of a table:

X \ Y	y_1	y_2	$\cdots$	y_m	Sum
x_1	$h(x_1, y_1)$	$h(x_1, y_2)$	$\cdots$	$h(x_1, y_m)$	$f(x_1)$
x_2	$h(x_2, y_1)$	$h(x_2, y_2)$	$\cdots$	$h(x_2, y_m)$	$f(x_2)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_n	$h(x_n, y_1)$	$h(x_n, y_2)$	$\cdots$	$h(x_n, y_m)$	$f(x_n)$
Sum	$g(y_1)$	$g(y_2)$	$\cdots$	$g(y_m)$	

The above functions f and g are defined by

$$f(x_i) = \sum_{j=1}^{m} h(x_i, y_j) \quad \text{and} \quad g(y_j) = \sum_{i=1}^{n} h(x_i, y_j)$$

i.e. $f(x_i)$ is the sum of the entries in the ith row and $g(y_j)$ is the sum of the entries in the jth column; they are called the *marginal distributions* and are, in fact, the (individual) distributions of X and Y respectively (Problem [illegible]). The joint distribution h satisfies the conditions

$$\text{(i)}\ \ h(x_i, y_j) \geq 0 \quad \text{and} \quad \text{(ii)}\ \ \sum_{i=1}^{n} \sum_{j=1}^{m} h(x_i, y_j) = 1$$

Now if X and Y are random variables with the above joint distribution (and respective means μ_X and μ_Y), then the *covariance* of X and Y, denoted by Cov (X, Y), is defined by

$$\text{Cov}(X, Y) = \sum_{i,j} (x_i - \mu_X)(y_j - \mu_Y)\, h(x_i, y_j) = E[(X - \mu_X)(Y - \mu_Y)]$$

or equivalently

$$\text{Cov}(X, Y) = \sum_{i,j} x_i y_j\, h(x_i, y_j) - \mu_X \mu_Y = E(XY) - \mu_X \mu_Y$$

The *correlation* of X and Y, denoted by $\rho(X, Y)$, is defined by

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Example : A pair of fair dice is tossed. We obtain the finite equiprobable space S consisting of the 36 ordered pairs of numbers between 1 and 6:

$$S = \{(1,1), (1,2), \ldots, (6,6)\}$$

Let X and Y be the random variables on S in Example 5.1, i.e. X assigns the maximum of the numbers and Y the sum of the numbers to each point of S. The joint distribution of X and Y follows:

$X \backslash Y$	2	3	4	5	6	7	8	9	10	11	12	Sum
1	$\frac{1}{36}$	0	0	0	0	0	0	0	0	0	0	$\frac{1}{36}$
2	0	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	0	0	$\frac{3}{36}$
3	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	0	0	$\frac{5}{36}$
4	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	0	0	$\frac{7}{36}$
5	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	0	0	$\frac{9}{36}$
6	0	0	0	0	0	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{11}{36}$
Sum	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	

The above entry $h(3,5) = \frac{2}{36}$ comes from the fact that $(3,2)$ and $(2,3)$ are the only points in S whose maximum number is 3 and whose sum is 5; hence

$$h(3,5) = P(X=3, Y=5) = P(\{(3,2), (2,3)\}) = \frac{2}{36}$$

The other entries are obtained in a similar manner.

We compute the covariance and correlation of X and Y. First we compute $E(XY)$:

$$\begin{aligned} E(XY) &= \sum x_i y_j h(x_i, y_j) \\ &= 1 \cdot 2 \cdot \tfrac{1}{36} + 2 \cdot 3 \cdot \tfrac{2}{36} + 2 \cdot 4 \cdot \tfrac{1}{36} + \cdots + 6 \cdot 12 \cdot \tfrac{1}{36} \\ &= \tfrac{1232}{36} = 34.2 \end{aligned}$$

By Example , $\mu_X = 4.47$ and $\mu_Y = 7$, and by Example 5.5, $\sigma_X = 1.4$ and $\sigma_Y = 2.4$; hence

$$\text{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = 34.2 - (4.47)(7) = 2.9$$

and

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{2.9}{(1.4)(2.4)} = .86$$

Example 5.7: Let X and Y, and X' and Y' be random variables with the following joint distributions:

$X \backslash Y$	4	10	Sum
1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
3	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
Sum	$\frac{1}{2}$	$\frac{1}{2}$	

$X' \backslash Y'$	4	10	Sum
1	0	$\frac{1}{2}$	$\frac{1}{2}$
3	$\frac{1}{2}$	0	$\frac{1}{2}$
Sum	$\frac{1}{2}$	$\frac{1}{2}$	

Observe that X and X', and Y and Y' have identical distributions:

x_i	1	3
$f(x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Distribution of X and X'

y_i	4	10
$g(y_i)$	$\frac{1}{2}$	$\frac{1}{2}$

Distribution of Y and Y'

We show that $\text{Cov}(X,Y) \neq \text{Cov}(X',Y')$ and hence $\rho(X,Y) \neq \rho(X',Y')$. We first compute $E(XY)$ and $E(X'Y')$:

$$E(XY) = 1\cdot4\cdot\tfrac{1}{4} + 1\cdot10\cdot\tfrac{1}{4} + 3\cdot4\cdot\tfrac{1}{4} + 3\cdot10\cdot\tfrac{1}{4} = 14$$

$$E(X'Y') = 1\cdot4\cdot0 + 1\cdot10\cdot\tfrac{1}{2} + 3\cdot4\cdot\tfrac{1}{2} + 3\cdot10\cdot0 = 11$$

Since $\mu_X = \mu_{X'} = 2$ and $\mu_Y = \mu_{Y'} = 7$,

$$\text{Cov}(X,Y) = E(XY) - \mu_X\mu_Y = 0 \quad \text{and} \quad \text{Cov}(X',Y') = E(X'Y') - \mu_{X'}\mu_{Y'} = -3$$

5.2. Independent Random Variables

A finite number of random variables $X, Y, \ldots, Z$ on a sample space S are said to be *independent* if

$$P(X = x_i, Y = y_j, \ldots, Z = z_k) = P(X = x_i)\,P(Y = y_j) \cdots P(Z = z_k)$$

for any values $x_i, y_j, \ldots, z_k$. In particular, X and Y are independent if

$$P(X = x_i, Y = y_j) = P(X = x_i)\,P(Y = y_j)$$

Now if X and Y have respective distributions f and g, and joint distribution h, then the above equation can be written as

$$h(x_i, y_j) = f(x_i)\,g(y_j)$$

In other words, X and Y are independent if each entry $h(x_i, y_j)$ is the product of its marginal entries.

Example : Let X and Y be random variables with the following joint distribution:

X \ Y	2	3	4	Sum
1	.06	.15	.09	.30
2	.14	.35	.21	.70
Sum	.20	.50	.30	

Thus the distributions of X and Y are as follows:

x	1	2
$f(x)$	.30	.70

Distribution of X

y	2	3	4
$g(y)$	.20	.50	.30

Distribution of Y

X and Y are independent random variables since each entry of the joint distribution can be obtained by multiplying its marginal entries; that is,

$$P(X = x_i, Y = y_j) = P(X = x_i)\,P(Y = y_j)$$

for each i and each j.

Theorem : Let X and Y be independent random variables. Then:

(i) $E(XY) = E(X)E(Y)$,

(ii) $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$,

(iii) $\text{Cov}(X,Y) = 0$.

Part (ii) in the above theorem generalizes to the very important

Theorem : Let $X_1, X_2, \ldots, X_n$ be independent random variables. Then

$$\text{Var}(X_1 + \cdots + X_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n)$$

<Best Regards>